

**EVOLUTION, THROUGH PARTICIPATION IN A RESEARCH  
GROUP, OF MOZAMBICAN SECONDARY SCHOOL  
TEACHERS' PERSONAL RELATION TO LIMITS OF  
FUNCTIONS**

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**A thesis submitted to the Faculty of Science, University of the  
Witwatersrand, Johannesburg, in fulfilment of the requirements for the  
degree of Doctor of Philosophy**

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## **DECLARATION**

**I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.**

**Danielle Jeanne Georgette Huillet**

**11th day of October 2007**

## **ABSTRACT**

This study investigates the evolution of four Mozambican teachers' personal relation to the limit concept through their participation in a research group, where they looked at mathematical issues concerning several aspects of the limit concept. It shows how the teachers' early personal relation to limits was close to the Mozambican Secondary School institutional relation to this concept, and how it evolved to a more elaborate relation. This evolution is shown to be uneven and limited for some aspects of mathematics for teaching limits which require deep understanding of basic mathematical concepts.

This study also provides an analysis of the teachers-as-researchers movement, where teachers' research mainly focuses on pedagogical issues, taking the mastery of mathematical knowledge for granted. It illustrates the difficulties faced by a teacher when challenging, not only his pedagogical practice, but also the mathematical content of his teaching. It concludes that this kind of research should be centred on mathematics.

## **DEDICATION**

**In memory of my father Jean Huillet who passed onto me his thirst for studies and research.**

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## **NOMENCLATURE**

EMU: Eduardo Mondlane University

MfT: Mathematics for teaching

PU: Pedagogical University

TT5-6: Teacher training for Grades 5 and 6

TT10-11: Teacher training for Grades 10 to 11

# CHAPTER 1

## INTRODUCTION

## **1 Introduction**

In Mozambique, as in many other countries, teachers usually teach mathematics according to institutional routines. They first learn these routines in schools as students, and then from their lecturers or more experienced teachers in their teaching practice, during and after their training. Their personal relation to mathematics is shaped by the institutions where they learnt, and does not allow them to break off from institutional routines. My contention is that, in order to question usual teaching, they would need to learn mathematics in a different way.

The purpose of this study is to investigate how mathematics teachers' personal relation to a mathematical concept evolves through their participation in a research group. The concept chosen for the teachers' research is that of limits of functions.

In this chapter I explain my motives for investigating this topic, the context of teacher training in Mozambique, the purpose of this study, and provide an outline of the thesis. For this purpose, the chapter is divided as follows:

- 1.1. Context of the study
- 1.2. Background to the study
- 1.3. Research question
- 1.4. Outline of the thesis

### **1.1 Context of the study**

Mozambique became independent in 1975. Throughout the first two years of independence most teachers, who were Portuguese, left the country. Many Mozambican children and adults, who had never had access to education, started studying. As a result, Mozambique faced a huge educational problem: a lot of students, few teachers, and few schools. In order to solve the lack of secondary school teachers, the last two years of secondary school (Grades 10 and 11) were closed in 1977, and their students sent to teach in lower grades (usually Grades 5 and 6), or to Eduardo Mondlane University (EMU) in pre-university courses or teacher training courses.

Mathematics teachers' training was planned in three phases. The first phase included one year of teachers' training for the first two years of secondary school, Grades 5 and 6 (referred here as TT5-6, Teacher Training for Grades 5 and 6), followed by two years of teaching these grades in a secondary school. These teachers would then come back to EMU for a further two-year training as mathematics and physics teachers for Grades 7, 8 and 9 (referred here as TT7-9), followed by another two years of teaching at that level. The third phase would be two years of training as mathematics and physics teachers for Grades 10 and 11 (referred here as TT10-11), equivalent to an Undergraduate degree in mathematics and physics.

According to this plan, and during three years (1977 to 1979), some students were trained as mathematics teachers for Grade 5 and 6. TT7-9 and TT10-11 started simultaneously in 1980. TT7-9 included some of the teachers coming back from their teaching practice, as well as students from Grade 9. TT10-11 featured other students coming back from their teaching practice, as well as students who had completed the pre-university course.

Due to a lack of students concluding Grades 9 and 11, very few teachers were trained according to this model.

This program ended in 1982 with the creation of the Pedagogical University, whose main aim was training teachers. Up until 2005, mathematics teachers were trained in two-subject courses (mathematics and physics) over four years.

## **1.2 Background to the study**

As a lecturer at Eduardo Mondlane University, I noticed that students usually seemed to consider the limit of a function when  $x$  tends to a finite value or to infinity as a number that had no applications. The study of limits in secondary schools seemed to be restricted to calculations. However, the limit concept is a very powerful concept, which can be studied from different points of view and using several representations, and can be used to solve several kinds of problems. As a student in France, a long time ago, I remember using limits to sketch the graph of a function and also to solve tasks in the context of geometry or other sciences. Why were Mozambican students not familiar with applying the limit concept for solving problems?

At a bridging course for first year university students, I introduced some tasks linking limits with graphs. When I explained the tasks to a young Mozambican colleague teaching the same subject for the first time, he looked very interested and, at the end of our conversation, his response was “it is so nice!”. It was probably the first time he came across these kinds of tasks.

I also remember my first class at university as a student, when the lecturer asked us to write down the definition of the limit of a function. Very few students were able to do it, although by that time the  $\varepsilon$ - $\delta$  definition was part of the French secondary school syllabus. Why did students fail in defining such a powerful concept? It seems that specific problems arise with this concept and with its teaching in schools.

In 1998, I initiated a research project with a colleague, aiming to investigate the teaching and learning of limits of functions in Mozambican secondary schools. Using Chevallard’s anthropological theory of didactics (1992) as our theoretical framework, our plan was to look at the Mozambican secondary schools’ institutional relation to limits, and some secondary school teachers’ and secondary school students’ personal relation to limits. We were then going to build a didactical engineering unit (Artigue, 1992) for teaching limits in schools in a more elaborated way. This didactical sequence would be used in trials in a Mozambican secondary school.

We started with the study of Mozambican secondary school institutional relation to limits through the study of the syllabus, a former textbook for Grade 12 (Berquembauv, Cherbakov, Mozolevski, Evdoquimov, Gerdes & Alexandrov, 1981) and thirty national examinations for Grade 12 (from 1981 to 1997). We concluded that there was a contradiction between, on the one hand, the theoretical development of the limit concept presented in the syllabus and in the textbook and, on the other hand, what was expected from the students as indicated by the exams and the textbook tasks, which were mainly algebraic (Mutemba & Huillet, 1999: 315).

We then looked at secondary school teachers’ personal relation to limits through a questionnaire applied to secondary school teachers in Maputo (the capital city)

and Quelimane (north of Mozambique), and through some teachers' interviews in Maputo. The analysis of the questionnaires and interviews showed that these teachers' personal relation to these concepts were generally in accordance with the secondary school institutional relation, as reflected in the examinations and the textbook (Huillet & Mutemba, 2000).

My colleague then carried out research on students' conceptions about the limit concept through a Masters Degree dissertation (Mutemba, 2001) and I went on working with teachers.

In the meantime, I supervised a Pedagogical University student's Honours dissertation on the teaching of inequalities (Costa, 1998). This student interacted with a teacher in order to construct a didactical unit for the teaching of inequalities, using changes of settings (Douady, 1986). The teacher was very enthusiastic with the new method, but in the classroom he was unable to use it and returned to his usual way of teaching. This difficulty is consistent with the problem known in French literature as the problem of "Reproducibility of Didactical Situations" ("Reproductibilité des situations didactiques"; Artigue, 1992; Arsac, 1989; 1992b; Legrand, 1996). While a researcher is successful in using a didactical unit s/he constructed on his/her own, another teacher usually fails in using a method s/he is not confident with.

Legrand (1996) describes:

Situations designed with very precise epistemological and didactical intentions and which seem to be explicitly described, were transformed (even completely altered) when used by colleagues who only superficially share our epistemological or didactical points of view, and/or our socio-cultural or ethical preoccupations.

[Des situations que nous avons construites avec des intentions épistémologiques et didactiques très précises et apparemment très explicitement décrites, ont été transformées (à la limite complètement dénaturées) quand elles ont été reprises par des collègues qui ne partageaient que très superficiellement nos points de vue épistémologiques



ou didactiques, et/ou nos préoccupations socioculturelles ou éthiques  
(1996:276)]

According to Artigue (1992), Robert & Robinet (1989) had already pointed out that

a certain compatibility of conception between researchers, who devise an engineering, and the teachers who are going to experiment with it or try to use it, is necessary for the effective working of didactic transmission.  
(1992: 61)

In line with these mathematics educators, I therefore decided not to work with teachers through didactical engineering approach, but instead on their personal relation to limits.

A new question arose: What could lead teachers to change their personal relation to limits of functions and the teaching of this concept?

Our previous research showed that Mozambican teachers' personal relation to limits was shaped by Mozambican didactic institutions' relation to this concept. This could be analysed in two ways. On the one hand, their knowledge about the limit concept would be limited to what they learnt about this concept in schools or during their teacher training. Few teachers had access to other sources of information, for example textbooks from other countries. On the other hand, the school tradition of teaching limits would strongly influence their view about the teaching of this topic in schools. It also was to be expected that the weight of the institutional relation be stronger for experienced teachers, as reported by other researchers. For example, Farah-Sarkis (1999) relates that experienced Lebanese teachers show a strong resistance to new teaching methods. She describes these teachers as "those with long years of teaching experience who were extremely comfortable with their practice, and who believed that knowledge of the subject matter area is all that they need" (1999:44). Mozambican teachers could be similar to Lebanese teachers.

The research concluded that teaching limits in a different way was only possible if teachers' knowledge about limits evolved, and if they were able to challenge the usual way of teaching limits in Mozambican secondary schools. This could occur

if they were in contact with limits through a new institution where this concept lived in a different way.

Which kind of institution could I set up for this purpose?

Several research papers have argued that teachers learn through research and through interaction within a community (Adler (1992; Mousley, 1992; Crawford & Adler, 1996; Zack, Mousley & Breen, 1997; D'Ambrosio, 1998; Jaworski, 1998). Moreover, teacher training at the Pedagogical University in Maputo included a dissertation about a research topic; and a Masters Degree in Mathematics Education had been recently set up at Eduardo Mondlane University, including a research project and dissertation. This led me to the idea of building a community of teachers researching different aspects of the limit concept under my supervision and sharing their findings.

This was the starting point for this study.

### **1.3 Research question**

The main question to be addressed by this study is:

How does teachers' personal relation to limits of functions evolve through their participation in a research group?

This main question needed to be refined by some sub-questions:

- What kind of knowledge does a teacher need in order to teach limits of functions in schools in a way different from the institutional relation to limits?
- What was the teachers' mathematical knowledge of limits of functions prior to the research?
- What ideas did they have of teaching this concept in schools prior to the research?
- What was their mathematical knowledge of limits of functions at the end of the research process?
- What ideas did they have of teaching this concept in schools at the end of the research process?

- For which aspects of limits did their knowledge evolve, for which aspects did their knowledge not evolve? How can this evolution be explained?

## 1.4 Outline of the thesis

In this chapter I have explained the background and the purpose of the study.

In Chapter 2, I present Chevallard's theory of didactic transposition (1985, 1991) and anthropological theory of didactics (1992, 1999), which is the theoretical framework which supports this study. I then provide an analysis of the reference mathematical organisation for teaching limits of functions, an analysis of the Mozambican secondary schools' relation to limits of functions, and an analysis of Pedagogical University's relation to limits of functions using this framework. I finally suggest possible ways of expanding the institutional relation to limits of functions in Mozambican secondary schools and the resultant mathematical challenges that a teacher would possibly face to do that.

In Chapter 3, I present and discuss the framework developed by Even (1990, 1993) and based on the notions of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) introduced by Shulman (1986, 1987), analyse the knowledge that a teacher needs to teach a specific mathematical topic in schools. I show that this framework presents several inconsistencies and then look at it through the lens of Chevallard's theories. This leads me to elaborate a new framework for *Mathematics for Teaching* (MfT) a specific topic, based on the knowledge that a teacher needs in order to consciously provide and support the didactical transposition.

In Chapter 4, I analyse *Mathematics for Teaching* limits of functions in the context of Mozambican secondary school. This analysis is based on a review of literature in the field, conceptually organised according to my categories of MfT a specific concept developed in Chapter 3. This also provides an idea of what ought to be the relation to the limit concept of the institution to be created with the aim of making teachers' knowledge evolve.

In Chapter 5, I present and discuss some experiences of *Teachers as Researchers*, in Mathematics Education, showing that the knowledge produced by this kind of

research is basically pedagogical knowledge. In most of the cases this does not lead teachers to challenge the institutional relation to mathematical knowledge, which is taken for granted. I then explain how teachers' activities within a new institution are expected to change their personal relation to the limit concept, and I describe how this institution, the research group, has been set up.

In Chapter 6, I describe and explain the methodology used in this study, regarding the creation of the institution, as well as data collection and analysis. I also discuss issues of validity and ethics in this research.

In Chapter 7, I provide a detailed analysis of the evolution of teachers' personal relation to limits according to one of my categories of MfT limits of functions: the organisation of students' first encounter with this concept. This analysis clearly establishes that teachers' personal relation to limits with regard to the first encounter did evolve during the research process. It also shows that challenging their own teaching is much more difficult for teachers than challenging the institutional relation.

In Chapters 8 and 9, I show that the teachers' personal relation to limits with respect to two additional aspects of MfT limits also evolved substantially, namely the social justification for teaching limits in schools and the essential features of the limit concept respectively.

In Chapter 10, I provide an analysis of the evolution of teachers' personal relation to the use of graphs in the study of limits. This analysis shows that the evolution of teachers' mathematical knowledge in this category was more restricted. I argue that working in the graphical register requires a deep understanding of basic mathematical concepts. Learning to use the graphical register through research was not effective because of a lack of understanding of the basic concepts involved. This case suggests that a more direct teaching should take place.

In Chapter 11, I analyse the evolution of teachers' personal relation to another aspect of the limit concept which also involves strong conceptual understanding of mathematics: the  $\varepsilon$ - $\delta$  definition. This analysis is consistent with the analysis made in the previous chapter. I thus conclude that for those aspects of MfT limits that

need strong mathematical knowledge, more direct teaching is needed in order to overcome this lack of conceptual understanding.

In Chapter 12, I draw out the main conclusions from the study, particularly as it relates to Mathematics teacher education in Mozambique, and discuss the limitations of this research. I also reflect on the results, the theoretical tools and the methodology.

The literature review of this thesis is not concentrated in a specific chapter, but distributed throughout Chapter 3 (literature on Mathematics for teaching), Chapter 4 (literature on limits of functions) and Chapter 5 (literature on teachers as researchers).

# **CHAPTER 2**

## **THE RELATION OF MOZAMBICAN DIDACTIC INSTITUTIONS TO THE LIMIT CONCEPT**

## 2 The Relation of Mozambican Didactic Institutions to the Limit Concept

As explained in the introduction, at the beginning of the research, I already had a view of the Mozambican secondary school's relation to the limit concept, from my experience of teaching at university and training secondary school teachers, as well as from a previous study conducted with a colleague.

That study (Mutemba & Huillet, 1999) presents an analysis of the Mozambican secondary schools' relation to the limit concept through the syllabus, a textbook and national examination. It concludes that there is a contradiction between, on the one hand, the theoretical development of the limit concept presented in the syllabus and the textbook and, on the other hand, what is expected from the students as indicated by the exams and the textbook's tasks. (1999: 315)

In fact, most of the tasks from 30 final national examinations from 1981 to 1997 were algebraic (93%), while only 4 of them (7%) were reading limits from a graph. Most of the textbook's tasks were also algebraic. We also reached the conclusion that the link between the algebraic register and the graphical register was very weak, and that secondary school students were basically required to solve algebraic tasks, without further applications.

A study of the personal relation of some secondary school Mozambican teachers to the limit concept, through a questionnaire and interviews (Huillet & Mutemba, 2000), showed that their personal relation was consistent with the secondary schools' institutional relation.

This relation has two main components:

- a formal one, derived from mathematical reasons which induces the teacher to teach the formal definition of limits despite awareness that the students do not understand it and are not able to apply it;
- an algebraic one where the students are asked to calculate indeterminate forms of limits, mainly because of the national exam tasks. This is exemplified by one teacher's comment: "*because at the exam it is only*

*calculations, yes it is only calculations, so we teach more calculations".*

(2000:195)

These previous studies were based on Chevallard's theory of institutional and personal relation to an object of knowledge (Chevallard, 1992). In the meantime, in further developments of his theories, Chevallard introduced more systematic tools to analyse an institutional relation (Chevallard, 1999; Bosch & Chevallard, 1999).

These tools have recently been used to analyse the teaching of limits of functions in Spanish High Schools (Barbé, Bosch, Espinoza & Gascón, 2005), starting from the description of the "reference mathematical organisation". This analysis is consistent with our study of Mozambican secondary school's relation to limit of functions (Mutemba & Huillet, 1999), which I revisit in this chapter using these new tools. As many secondary school teachers have been trained at the Pedagogical University, I also analyse here this institution's relation to limits of functions.

Finally I suggest possible ways of expanding the institutional relation to limits of functions in Mozambican secondary schools. Considering the institutional constraints presented in this chapter, I then analyse the resultant difficulties that could be faced by the teachers in this study.

This chapter is therefore structured as follows:

- 2.1. Chevallard's anthropological theory of didactics
- 2.2. The reference mathematical organisation
- 2.3. Mozambican secondary schools' relation to limits of functions
- 2.4. The Pedagogical University's relation to limits of functions
- 2.5. Conclusion

## **2.1 Chevallard's anthropological theory of didactics**

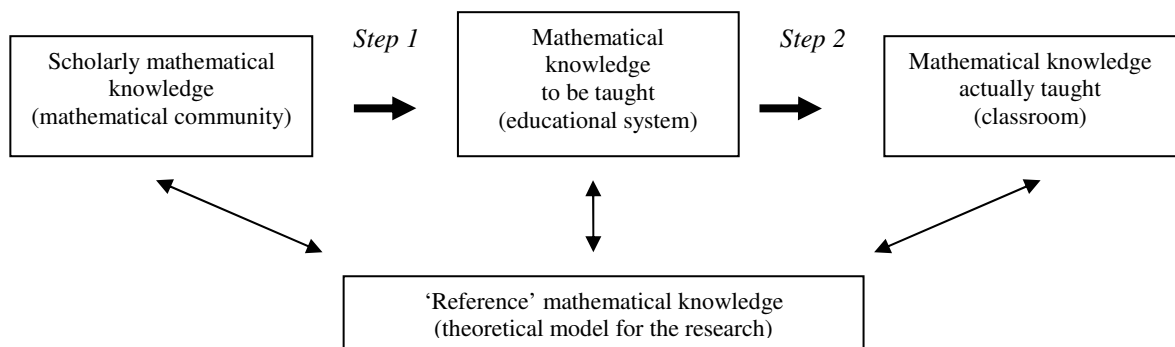
The anthropological theory of didactics has its roots in the theory of didactical transposition, which first appears in the work of Chevallard in 1985. I will present the evolution of Chevallard's theories, considering three main periods: the theory of didactical transposition (Chevallard, 1985), the first anthropological approach,



that is institutional and personal relations (Chevallard, 1992), and the theory of praxeological organisations (Chevallard, 1999).

### 2.1.1 The didactical transposition

The theory of didactical transposition points out the difference between school mathematics and scholarly mathematics and offers a model of the process through which a mathematical object is converted into an object that can be taught in school institutions. Chevallard (1985) distinguishes two main steps during the process, that Barbé et al. represent as follows<sup>1</sup> (2005: 241):



**Figure 2.1 The didactical transposition**

The first step of the didactical transposition (Scholarly mathematical knowledge → Mathematical knowledge to be taught) consists in identifying, from the works of mathematicians, objects that ought to be taught in schools. In the second step (Mathematical knowledge to be taught → Mathematical knowledge actually taught), these objects are then transformed into objects of teaching, according to the age of the learners and to institutional constraints.

In that way, the knowledge produced by the didactical transposition is

exiled from its origins, and cut off from its historical production in the scholars' knowledge sphere<sup>2</sup>.

[exilé de ses origines, et coupé de sa production historique dans la sphère du savoir savant.] (Chevallard, 1991: 17)

<sup>1</sup> "Step 1" and "Step 2" in Barbé et al. diagram added by author

<sup>2</sup> All quotes in French or in Portuguese have been translated by author

For example the epistemological study of the concept of limits of function shows that it has been developed by mathematicians over many centuries. This development has been guided by the necessity of solving different kinds of problems, and is the result of many debates between mathematicians.

Nevertheless the concept of limits as taught in most school institutions appears as if it were a stable body of knowledge. All difficulties, hesitations and mistakes faced over time by the mathematicians when conducting their research about this concept have been eliminated.

In an analysis of didactical transposition, Arsac (1992a) points out two main specificities of this theory.

The theory of didactic transposition brings two fundamental points to light:

- the problem of justifying the contents of teaching,
- the systematic appearance of a gap between taught knowledge and the references that legitimate it, a gap due to the constraints weighing on how the teaching system functions. (1992: 108)

The publication of Chevallard's work about the didactical transposition in 1985 gave rise to much criticism and debate, which has been clearly summarised by Arsac (1992a). I will not go into the details of all discussion that arose at that time but I want to highlight one of these points, as it has implications for my study:

Where does the cognitive lie?

Unlike many didactic theories of the 1980s, the transposition theory does not seem to take into account the cognitive mechanisms of learning. Arsac explains this fact arguing that

the theory was mainly used at the beginning for studying phenomena which take place before the teacher's, and therefore the pupil's work, which is enough to highlight the phenomenon of transposition (...) (1992a: 119)

According to Bosch & Chevallard (1999)

Its main contribution was not only to evidence the distance between scholarly knowledge and knowledge to be taught, and thus the necessary transformations of a mathematical object in order to be taught, but, above all, the didactical

transposition points out that the *mathematical knowledge is the starting-point of any didactical problematic*.

[son apport principal n’a pas été seulement de mettre en évidence la *distance* qui sépare le savoir savant du savoir à enseigner, et donc les transformations nécessaires que doit subir tout objet mathématique pour pouvoir être enseigné. Ce que montre surtout la notion de transposition didactique, c’est que le *savoir mathématique* est à l’origine de toute *problématicité didactique*.] (1999: 82)

While didactical transposition theory looks mainly at the knowledge to be taught and at the process of its production, further developments of Chevallard’s theories give more room to the actors of the didactic situation: teachers and learners. As my purpose is to look at teachers’ knowledge, I will describe these theories in more detail, starting with the first notions of the anthropological approach: the institutional and personal relation to a concept.

### **2.1.2 The first anthropological approach: institutional and personal relation**

In the epilogue of the second edition of the theory of didactical transposition, Chevallard locates didactic phenomena in the field of anthropology (1991: 205). This point of view is developed in further publications (Chevallard 1992, 1995, 1999; Bosch & Chevallard, 1999).

In a rather axiomatic way, Chevallard (1992) presents as primitive terms of his theory the notions of objects, persons, and institutions.

He considers that “everything is an object” and that an object exists as soon as a person or an institution recognises this object as existing, if at least one person or one institution relates to this object (1992: 142).

The word “institution” is used in the broader sense of the term: it can be a school, a class, but also “practical work”, “lectures”, “family” and others (1992: 144). A set of “institutional objects” is associated to each institution, objects for which an institutional relation, with stable elements, those that appear “self-evident, transparent, non-problematic” exists for the subjects of the institution (1992: 145).

When a person enters an institution s/he becomes a subject of this institution. An institutional object “comes to life” for this person under the constraint of the institutional relation. This object may or may not have existed for this person before. A personal relation will change, or will be constructed. There is learning (Chevallard, 1992: 145-46).

Chevallard also introduces the notion of “good subject” of an institution: a person becomes a “good subject” of an institution in relation to an institutional object when his/her personal relation to this object is judged to be *consistent* with the institutional relation (1992: 146).

Didactic institutions are particular institutions which include one or more subject (usually teachers and learners) and a set of didactic objects. Their purpose is to transform the personal relation of each subject to these objects. The aim is that this relation is consistent with the institutional relation (1992: 146-47).

Nevertheless, a didactic system never exists alone, but together with other didactic systems which influence, on the one hand the functioning of this particular system, and on the other hand the personal relation of an individual to an object of knowledge. As a person is a subject of numerous institutions, Chevallard argues that s/he is the “emergent complex web of institutional subjections” (1992: 147). He adds:

What we call the “liberty” of a person thus appears as the effect obtained by playing off one or more institutional subjections against each other. (1992: 147)

In addition, the personal relation to an object can include a public component (relative to an institution), and a private component, which escapes evaluation in that institution. The part of the personal relation which does not appear in one institution can become visible in full light in another institution (1992: 147).

Let’s look at the special case of limits of functions and Mozambican mathematics teachers. In line with Chevallard, my argument is that Mozambican mathematics teachers’ relation to the limit concept has been shaped by the relation to this concept of the institutions where they met it. For most of them, this contact has occurred in Mozambican institutions (secondary school as students, university as students, and secondary school as teachers). The relation to the limit concept of these didactic

institutions needs to be analysed, in order to better understand the teachers' personal relation to this concept.

How can we analyse the institutional relation to a concept?

According to Chevallard, the institutional relation to an object of knowledge can be analysed through the social practices involving this object inside the institution. He elaborates a method to describe and analyse these institutional practices, using the notion of praxeological organisation or praxeology.

### 2.1.3 The praxeological organisations

The first assumption of the theory of praxeological organisations is that

any institutional practice can be analysed from different points of view and in different ways, as a system of tasks.

[toute pratique institutionnelle se laisse analyser, de différents points de vue et de différents façons, en un système de tâches.] (Bosch & Chevallard, 1999: 84)

Mathematics, as a human activity, is not an exception to the rule.

As a second assumption, Chevallard states that

Inside a given institution I, around a specific kind of task T, there is generally only one technique, or at least very few techniques recognized by the institution.

[en une institution I donnée, à propos d'un type de tâches T donné, il existe en général une seule technique, ou du moins un petit nombre de techniques institutionnellement reconnues.] (1999 : 225)

Each kind of task and the associated technique form the *practical bloc* (or know-how) of a *praxeology*, also called *praxeological organisation* or, in the case of mathematics, *mathematical organisation* (MO).

For example, in Mozambican secondary schools, students are taught to calculate limits using algebraic transformations. A specific algebraic transformation is associated to each kind of limit, constituting the practical block of a specific MO. Other kinds of tasks could be: to read limits from a graph, to sketch the graph of a function using its limits, to demonstrate the limit of a function using the definition,

etc. Students would be expected to solve each of these tasks using a specific technique.

The institutional relation to an object is shaped by the set of tasks to be performed, using specific techniques, by the subjects holding a specific position inside the institution. In an institution, a specific kind of task is usually solved using only one technique. Most of the tasks become part of a routine, the task/technique practical blocks appearing to be *natural* inside this institution.

For example, the task “calculate  $\lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1}$ ” is usually solved in Mozambican secondary schools by using the following technique:

$$\lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1} = \lim_{x \rightarrow +\infty} \frac{x^2 \left(1 + \frac{2}{x^2}\right)}{x^2 \left(1 + \frac{1}{x^2}\right)} = \lim_{x \rightarrow +\infty} \frac{1 + \frac{2}{x^2}}{1 + \frac{1}{x^2}} = \frac{1 + 0}{1 + 0} = 1.$$

This technique seems natural to teachers and they do not question its validity or efficiency. Other techniques to solve the same task could be the following ones:

$$\lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1} = \lim_{x \rightarrow +\infty} \frac{\frac{x^2 + 2}{x^2}}{\frac{x^2 + 1}{x^2}} = \lim_{x \rightarrow +\infty} \frac{1 + \frac{2}{x^2}}{1 + \frac{1}{x^2}} = \frac{1 + 0}{1 + 0} = 1, \text{ being a small variation of}$$

the first technique;

$$\lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1} = \lim_{x \rightarrow +\infty} \frac{x^2}{x^2} = 1, \text{ which is based on approximations;}$$

$$\lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1} = \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x^2 + 1}\right) = 1 + 0 = 1, \text{ by dividing the fraction;}$$

$$\text{or } \lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1} = \lim_{x \rightarrow +\infty} \frac{2x}{2x} = 1, \text{ using L'Hôpital's Rule.}$$

In general, these latter four techniques are not institutionally recognised in Mozambican secondary schools.

The third assumption of the theory of praxeological organisations is that there is an ecological constraint to the existence of a technique inside an institution: it must appear to be understandable and justified (Bosch & Chevallard, 1999: 85-86). This is

done by the technology, which is a rational discourse to describe and justify the technique. This constraint can be interpreted at two levels. At students' level, it means that students should be able to understand the technique. At mathematics level, we must ensure that the technique is “mathematically correct”, in reference to the scholarly knowledge. These ecological constraints can sometimes lead to a contradiction, given that the ability of students to understand will be constrained by their age and previous knowledge. It can be difficult for a technique to be both understandable and justified at the same time.

Elements of the technology can be integrated in the technique. For example, when teaching the institutionally recognized technique to calculate the limit  $\lim_{x \rightarrow +\infty} \frac{x^2 + 2}{x^2 + 1}$ , a

Mozambican teacher would explain:

$$\lim_{x \rightarrow +\infty} \frac{x^2 \left(1 + \frac{2}{x^2}\right)}{x^2 \left(1 + \frac{1}{x^2}\right)} \text{ “taking out the highest power of } x \text{ as common factor in the numerator and in the denominator”}$$

$$\lim_{x \rightarrow +\infty} \frac{1 + \frac{2}{x^2}}{1 + \frac{1}{x^2}} \text{ “simplifying the fraction”}$$

$$\frac{1+0}{1+0} = 1 \text{ “substituting each limit by its value”}.$$

The explanation that comes with each step of the solution is part of the technology. Another part of the technology for this technique would be the theorems about limits such as:

$$\lim_{x \rightarrow \infty} [f(x) + g(x)] = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x), \text{ if } \lim_{x \rightarrow \infty} f(x) \text{ and } \lim_{x \rightarrow \infty} g(x) \text{ exist.}$$

$$\lim_{x \rightarrow \infty} [f(x) \cdot g(x)] = \lim_{x \rightarrow \infty} f(x) \cdot \lim_{x \rightarrow \infty} g(x), \text{ if both limits exist.}$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow \infty} f(x)}{\lim_{x \rightarrow \infty} g(x)}, \text{ if } \lim_{x \rightarrow \infty} f(x) \text{ and } \lim_{x \rightarrow \infty} g(x) \text{ exist, and } \lim_{x \rightarrow \infty} g(x) \neq 0.$$

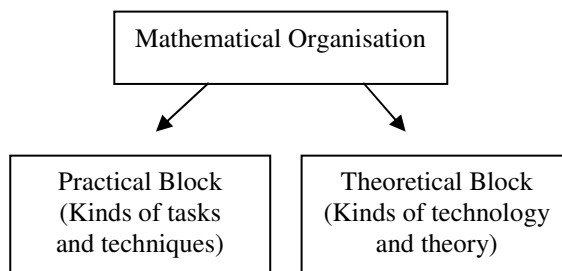
If  $r$  is a positive rational number and  $c$  is any real number, then  $\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0$

The technology itself is justified by a theory, which is a higher level of justification, explanation and production of techniques. For instance, in the above example, the institutionalized technology could be justified by the demonstration of the theorems about limits, using the  $\varepsilon$ - $\delta$  definition.

Technology and theory constitute the *knowledge block* of a MO. According to Chevallard, the technology-theory block is usually identified with *knowledge* [un savoir], while the task-technique block is considered as *know-how* [un savoir-faire] (1999: 228). This explains the use of the term *praxeology*.

The word ‘praxeology’ indicates that practice (*praxis*) and the discourse about practice (*logos*) always go together, even if it is sometimes possible to find local know-how which is (still) not described and systematised, or knowledge ‘in a vacuum’ because one does not know (or one has forgotten) what kinds of problems it can help to solve. (Barbé et al., 2005: 237)

The two components of an MO are summarized in the diagram below.



**Figure 2.2 Mathematical Organisation**

A MO around a specific kind of task in a specific institution, such as the one described above, is a specific one. The integration of several specific MOs around a specific technology gives rise to a local MO. For example, calculating limits using algebraic transformation in Mozambican secondary schools constitutes a local MO. In the same way, the integration of several local MOs around the same theory gives rise to a regional or global MO.

In order to understand teachers’ personal relation to limits of functions, it is important to analyse the relation to limits of the institutions where they have met this concept. Most of them met limits in Mozambican secondary schools, as students and/or as teachers, and at the Pedagogical University. Therefore, I have considered



these two institutions as representative of the institutions which have shaped their personal relation to limits. We will see later that other institutions where some of the teachers met the limit concept (Faculty of Education at EMU, or the Pedagogical Institute in Germany) had very similar institutional relation.

Nevertheless, before doing this analysis, we need to analyse the reference mathematical organisation in scholarly mathematical knowledge.

## 2.2 The reference mathematical organisation

The first step of the didactical transposition consists in identifying content in *scholarly mathematical knowledge* and converting it into *knowledge to be taught*. In the case of limits of functions, Barbé et al. structured this knowledge as a *reference mathematical organisation* “which includes and integrates in a regional organisation two different local organisations  $MO_1$  and  $MO_2$  that will assume different roles” (2005: 241).

The first mathematical organisation,  $MO_1$ , can be named the algebra of limits. It starts from the supposition of the existence of the limit of a function and poses the problem of how to determine its value – how to calculate it – for a given family of functions. Two main types of problems or problematic tasks  $T_1$  of  $MO_1$  are as follows:

$T_1$ : Calculate the limit of a function  $f(x)$  as  $x \rightarrow a$ , where  $a$  is a real number.

$T_2$ : Calculate the limit of a function  $f(x)$  as  $x \rightarrow \pm\infty$ .

In both cases the function  $f(x)$  is supposed to be given by its algebraic expression and the techniques used to calculate the limits are based on certain algebraic manipulations of this expression (factoring, simplifying, substituting  $x$  by  $a$ , etc.). (2005: 241)

They also include in  $MO_1$  a third kind of task, even if it is not an algebraic task.

$T_3$ : Determine the limit of a function given its Cartesian graph  $y = f(x)$ . (2005: 242)

This kind of task has been included in  $MO_1$  because it appears closely related to the algebraic tasks, and even subordinate to them.

Finally, the fourth kind of task belonging to  $MO_1$  relates to continuity of functions.

**T<sub>4</sub>:** Study the continuity of  $f(x)$ . (2005: 244)

$T_4$  tasks are subordinate to the first three types of tasks, because they usually require algebraic manipulations to calculate one-sided limits ( $T_1$ ) or reading limits from the graph ( $T_3$ ).

The technology needed to justify these kinds of tasks and techniques, as illustrated in the first part of this chapter, consists, on the one hand, in justifying algebraic transformations and, on the other hand, in the algebra of limits. An example of some of these properties (limit of a sum, product, quotient of functions, squeeze theorem) presented by to Larson, Hostetler & Edwards (1994) can be found in Appendix 2.1. All these properties aim to justify the techniques used to calculate limits. They are part of the technology. At a more theoretical level, these rules can be justified by the theory of real numbers.

The second mathematical organisation  $MO_2$  addresses the existence of limits.

This mathematical organisation emerges from the question of the nature of the mathematical object ‘limit of a function’ and aims to address the problem of the *existence of limit* with respect to different kinds of functions. (1994: 242)

They indicate the following types of problematic tasks as belonging to  $MO_2$ :

**T<sub>6</sub><sup>3</sup>:** Prove the existence (or non-existence) of the limit of a function  $f$  as  $x \rightarrow a$ , where  $a$  is a real number, or  $x \rightarrow +\infty$ .

**T<sub>7</sub>:** Prove the existence (or non-existence) of one-sided limits for certain kinds of functions (such as monotonic functions).

**T<sub>8</sub>:** Prove the properties used to justify the way certain limits of functions are calculated.

The “knowledge block” for  $MO_2$  is mainly based on the  $\varepsilon$ - $\delta$  definition of limits and, at a deeper level, on the theory of real numbers.

Although these two local organisations  $MO_1$  and  $MO_2$  could appear to be separate within the reference organisation for limits of functions,  $MO$ , they are in fact closely

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<sup>3</sup> A 5<sup>th</sup> kind of task, belonging to  $MO_1$ , will be introduced later.

related. On the one hand, the technology of  $MO_1$  is partly contained in  $MO_2$ , in terms of the proof of the rules used to calculate limits, on the other hand, they share the same theory of real numbers, even if this theory is not visible in  $MO_1$ .

This reference mathematical organisation of limits of functions has been described by Barbé et al. (2005) as an epistemological model of the “scholarly knowledge” that legitimates the “knowledge to be taught” in Spanish high schools. I will use this model as a base to analyse the institutional relation to the limit concept of Mozambican secondary school and of the Pedagogical University, adapting it to the Mozambican context when necessary.

### **2.3 Mozambican secondary schools’ relation to limits of functions**

As I said before, I will revisit here the analysis of Mozambican secondary schools’ relation to limits done some years ago (Mutemba & Huillet, 1999) with the new tools introduced by Chevallard (1999) in his theory of praxeological organisation and the reference mathematical organisation of limits as described by Barbé et al. (2005).

Traditionally, Mozambican teachers use the syllabus, the final national examinations and worksheets to prepare their lessons. There was no official textbook for the two last years of high school until 2005<sup>4</sup> (Grades 11 and 12). The only textbook produced in Mozambique for this level until 2005 was written in 1981 by lecturers of Eduardo Mondlane University (Berquembauve et al., 1981) but it has been out of print for many years now. As our previous study about the personal relation of secondary school mathematics teachers to the limit concept (Huillet & Mutemba, 2000) showed that most of them did not use this textbook, I did not consider it relevant for this study. Some mathematics teachers also use Portuguese textbooks, but not all of them can access these materials.

Therefore, I used three main sources to describe the Mozambican secondary schools’ relation to limits of functions: the syllabus, the final national

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<sup>4</sup> A textbook for Grade 11 was edited in 2006, a textbook for Grade 12 will be published in 2007

examinations and two worksheets about limits produced by teachers of the two main high schools in Maputo.

The syllabus and the final examinations, as official documents, relate to the *knowledge to be taught*, which results from the first step of the didactical transposition (Scholarly mathematical knowledge  $\rightarrow$  Mathematical knowledge to be taught). Nevertheless, as there is not a specific textbook for this level in Mozambique, experienced teachers of some high schools produced worksheets about limits of functions that are given to new teachers to prepare their lessons. These worksheets also serve as a reference of the *knowledge to be taught*.

To describe in detail the *knowledge actually taught*, it would be necessary to use other sources, such as students' exercise-books or classes' observations. As my purpose is only to have a view of the institutional relation to limits, in order to better understand teachers' personal relation to this concept, I will not extend the analysis in such detail. What I will describe here is basically the *knowledge to be taught*, as it can be seen at school level.

This analysis is divided into two parts: the practical block and the knowledge block.

### 2.3.1 The practical block (kinds of tasks/techniques)

#### *The syllabus*

The Mozambican syllabus (Ministério da Educação, 1997) is not very explicit about what kinds of tasks ought to be taught in secondary schools. Four of the objectives relate to tasks belonging to the local MO<sub>1</sub> as described in section 2 of this chapter.

- |  |   |
|--|---|
| - determinar o limite de uma função nos dois casos indicados no objectivo anterior [ $x \rightarrow a$ e $x \rightarrow \infty$ ]; | - to determine the limit of a function in the two cases indicated in the previous objective [ $x \rightarrow a$ and $x \rightarrow \infty$ ]; |
| - identificar as formas indeterminadas de limites de funções; levantar as indeterminações;   | - to identify indeterminate forms of limits of functions; to handle these indeterminate forms;  |
| - calcular limites laterais;   | - to calculate one-sided limits;  |
| - identificar, justificar e aplicar os limites   | - to identify, justify and apply the special  |

$$\begin{array}{ll} \text{notáveis: } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e; \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; & \text{limits: } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e; \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; \\ \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e. \text{ (1997: 31)} & \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e. \end{array}$$

The three first objectives above indicate what kinds of limits students are expected to calculate ( $x \rightarrow a$ ;  $x \rightarrow \infty$ ), but do not specify for what kind of functions. The chapter about sequences (Unit IV) is more explicit. One of the objectives of this chapter is:

$$\begin{array}{ll} \text{- calcular automaticamente limites simples,} & \text{- to automate the calculation of simple} \\ \text{tais como: } \lim_{u_n \rightarrow 0} \frac{1}{u_n}; \lim_{n \rightarrow \infty} p^n; \lim_{u_n \rightarrow \infty} \frac{1}{u_n}; & \text{limits, such as } \lim_{u_n \rightarrow 0} \frac{1}{u_n}, \lim_{n \rightarrow \infty} p^n, \\ \lim_{n \rightarrow \infty} \frac{P(n)}{Q(n)}. \text{ (1997: 30)} & \lim_{u_n \rightarrow \infty} \frac{1}{u_n}, \text{ and } \lim_{n \rightarrow \infty} \frac{P(n)}{Q(n)}. \end{array}$$

The methodological orientations for this unit IV reads:

$$\begin{array}{ll} \text{Os casos de indeterminação para este nível} & \text{The indeterminate forms for this level are:} \\ \text{são: } \frac{\infty}{\infty}; \frac{0}{0}; \infty - \infty; 0 \cdot \infty; 1^\infty. & \frac{\infty}{\infty}; \frac{0}{0}; \infty - \infty; 0 \cdot \infty; 1^\infty. \end{array}$$

Na unidade a seguir, serão retomadas as mesmas indeterminações, no caso de funções. (1997: 31)

In the next unit, the same indeterminate forms will reappear, in the case of functions.

From these statements, we can surmise that students are expected to calculate limits of functions that lead to the indeterminate forms indicated above.

Nevertheless, the difficulty of such indeterminate forms is not specified. For

example, both limits  $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1}$  and  $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x - 1)^2}$  lead to the indeterminate

form  $\frac{0}{0}$ , but it is easier to solve the first one. This shows that these objectives can be interpreted differently by different teachers.

It is also difficult to infer from the objectives what techniques should be used to solve these limits. The fourth objective is more explicit about the techniques to be used, the kinds of tasks to be solved being implicit: algebraic tasks which lead to

the use of the special limits  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ ,  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and  $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$ .

In any case, even if they are not explicit, all the tasks related to these four objectives can be classified as  $T_1$  or  $T_2$  belonging to  $MO_1$ .

Two of the objectives relating to continuity correspond to two new kinds of tasks about limits.

- |   |  |
|---|--|
| - identificar uma função contínua dado o seu gráfico;                             | - to identify a continuous function given by its graph;                                  |
| - determinar se uma função é contínua, dada a sua expressão analítica. (1997: 31) | - to determine whether a function given by its analytic expression is continuous or not. |

These objectives correspond to the kind of task  $T_4$ . The first one is also related to  $T_3$  and the second one to  $T_1$ . Nevertheless, the syllabus does not indicate what kind of functions should be analysed in terms of continuity.

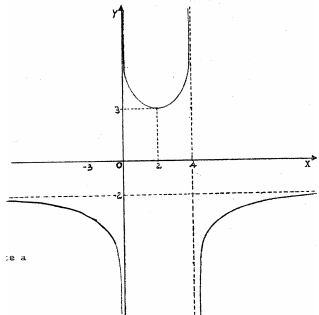
As the syllabus is not explicit about the practical block of the MO for limits of functions, it is to be expected that teachers would turn to the national examinations in order to prepare their lessons. In fact, to be “good subjects” of the Mozambican secondary school institution, they are expected to prepare their students for the national examination.

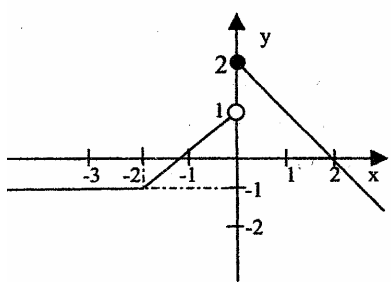
### ***The national examinations***

At the end of high school, students are required to write a national examination. There are generally two periods of examinations, the students who failed at the first period being allowed to attend the second period. There are usually two calls at the first period, the participation at the second one being restricted to students who were not able to attend the first call. This means that every year the Ministry of Education must set three examinations for Grade 12. In our previous study (Huillet & Mutemba, 2000), we analysed 30 national examinations for Grade 12 from 1981 to 1997. These examinations evidenced the same structure. In all of them, there is one task dedicated to the calculation of two or three limits, which generally lead to indeterminate forms. In some exams we also found a task to read limits from the graph or a task about the continuity of functions. A classification of these tasks was presented at that time (Huillet & Mutemba, 2000: 313). I will classify them again, using the reference mathematical organisation presented in section 2.2 of this chapter. I did not take into account the exams after 1997,

because they present the same structure and I assumed that they would not influence the results. The new classification can be seen in Table 2.1.

**Table 2-1 Number of tasks about limits in 30 final exams of secondary school**

<b>T<sub>1</sub></b>	<b>CALCULATE THE LIMIT OF A FUNCTION WHEN <math>x \rightarrow a</math></b>	
<b>T<sub>1A</sub></b>	Limit of a continuous function. Example: $\lim_{x \rightarrow 2}  x + 5 $ (Worksheet 2, p.7)	0
<b>T<sub>1B</sub></b>	Indeterminate form of a rational function when $x \rightarrow a$ Example: $\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 2x}{x^2 - 4x + 3}$ (1999/1 <sup>st</sup> period/1 <sup>st</sup> call)	13
<b>T<sub>1C</sub></b>	Indeterminate form involving square roots Example: $\lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$ (1997/1 <sup>st</sup> period/1 <sup>st</sup> call)	15
<b>T<sub>1D</sub></b>	Indeterminate form involving a trigonometric function Example: $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 3x}$ (1996/2 <sup>nd</sup> period)	19
<b>T<sub>2</sub></b>	<b>CALCULATE THE LIMIT OF A FUNCTION WHEN <math>x \rightarrow \pm\infty</math></b>	
<b>T<sub>2A</sub></b>	Limit of a polynomial, a rational or an irrational function when $x \rightarrow \pm\infty$ Examples: $\lim_{x \rightarrow +\infty} (-3x^3 + 4x - 1)$ (1988/2 <sup>nd</sup> period); $\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 - x}$ (1993/1 <sup>st</sup> period)	3
<b>T<sub>2B</sub></b>	Indeterminate form such as $1^\infty$ Example: $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n}\right)^n$ (1997/1 <sup>st</sup> period/2 <sup>nd</sup> call)	2
<b>T<sub>3</sub></b>	<b>DETERMINE THE LIMIT OF A FUNCTION FROM THE GRAPH</b> Example (1988/1 <sup>st</sup> period) The figure shows the graph of a function f. Observe the graph and answer: a) Determine the domain of f. b) Determine the range of f. c) Determine: $\lim_{x \rightarrow 2} f(x)$ $\lim_{x \rightarrow 4+0} f(x)$ $\lim_{x \rightarrow +\infty} f(x)$ 	4

<b>T<sub>4</sub></b>	<b>STUDY THE CONTINUITY OF A FUNCTION</b>	
<b>T<sub>4A</sub></b>	<p>Discussion of the continuity of a function using its analytical expression</p> <p>Example: Given a function <math>f</math> such as <math>f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x^2 - 9}, &amp; \text{if } x \neq 3 \\ m + 4, &amp; \text{if } x = 3 \end{cases}</math></p> <p>a) Is the function continuous for <math>x = -3</math>? Justify your answer.</p> <p>b) Determine the value of the parameter <math>m</math>, such that the function <math>f</math> is continuous for <math>x = 3</math> (1992/1<sup>st</sup> period)</p>	6
<b>T<sub>4B</sub></b>	<p>Discussion of the continuity of a function using its graph</p> <p>Observe the graph, which represents a function <math>f</math>.</p> <p>a) Determine:</p> <ul style="list-style-type: none"> <li>• <math>f(0)</math></li> <li>• <math>\lim_{x \rightarrow 0^-} f(x)</math></li> <li>• <math>\lim_{x \rightarrow 0^+} f(x)</math></li> </ul> <p><math>f'(-2)</math> (2001/1<sup>st</sup> period/1<sup>st</sup> call)</p> 	4

In this classification, I subdivided  $T_1$  into four kinds of tasks ( $T_{1A}$ ,  $T_{1B}$ ,  $T_{1C}$ , and  $T_{1D}$ ) and  $T_2$  into two kinds of tasks ( $T_{2A}$  and  $T_{2B}$ ), because they correspond to different techniques.  $T_{1A}$  does not occur in the examinations but will appear later in the worksheets. I included it here because it is the only new kind of task that will appear in the worksheets.  $T_4$  has also been subdivided into two kinds of tasks ( $T_{4A}$ ,  $T_{4B}$ ), the first one being an algebraic task and the second one a graphical task. For each kind of task I inserted inside the table an example from an exam with its reference (year/period/call). The last column of Table 2.1 indicates the number of occurrences of each kind of task within the 30 examinations.

This table clearly shows that most of the tasks about limits from the examinations were algebraic, which I classified as  $T_1$  (71%),  $T_2$  (8%) and  $T_{4A}$  (9%), while only 8 of them were graphical tasks:  $T_3$  (6%) or  $T_{4B}$  (6%). All of them belong to  $MO_1$ . From these tasks, we can surmise that secondary school teachers are mainly expected to teach algebraic techniques to calculate indeterminate forms of limits and, to a lesser extent, to read limits from graphs.

The technique to be taught for each kind of task is presented in Appendix 2.3.



**Worksheet 1 (WS1)**

This worksheet (see Appendix 2.4) comes from Josina Machel Secondary School and is entitled *Limit of functions and continuity* [Limite de funções e continuidade]. It is divided into five sections.

Section I includes 23 tasks to calculate the limit of a rational or irrational function when  $x$  tends to infinity. They belong to  $T_{2A}$ , but most of them are much more complicated than the limits that can be found in the national examinations, for

example  $\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}}$  (number 13) or  $\lim_{x \rightarrow \infty} x(\sqrt{x^2 - 5x + 6} - x)$  (number 20).

Section II presents 13  $T_{1A}$ , 12  $T_{1B}$  and 25  $T_{1C}$  tasks to calculate limits. As for section

I, some of these tasks are quite difficult, as for example  $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x-1)^2}$  (number

36). Section III presents 15 limits involving the special trigonometric limit

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ . For example  $\lim_{x \rightarrow 0} \frac{x + \sin 3x}{4x - \tan 2x}$  and  $\lim_{x \rightarrow 0} \frac{\sqrt{2x+1} - \sqrt{x+1}}{\sin x}$ . I classified

them as  $T_{1D}$ , even if several techniques need to be used to solve some of them. For example the last task is also a  $T_{1C}$  task.

Section IV is dedicated to 7 limits leading to indeterminate forms with exponents

such as  $[1^\infty]$ . In particular, some of them lead to the special limit  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ ;

others involve trigonometric functions or logarithms. I classified them as  $T_{2B}$ .

Section V is divided into three tasks.

Task 1 asks to calculate limits of 7 functions when  $x \rightarrow +\infty$  and when  $x \rightarrow -\infty$ : two simple rational functions, one irrational function, and 4 simple exponential functions. All of them have been considered as  $T_{2A}$  tasks.

Task 2 is dedicated to one-sided limits of 5 functions. I classified them, according to the kind of function involved, as 1  $T_{1A}$ , 3  $T_{1B}$  and 1  $T_{1D}$ .

Task 3 reads “Study the continuity of the following functions and classify the point(s) of discontinuity” [Estude a continuidade das seguintes funções e classifique

o(s) ponto(s) de descontinuidade]. They are all polynomial or rational functions, 6 of them being piece-wise functions. The limits to be calculated in section V are quite simple, compared to the tasks in the previous sections. They are T<sub>4A</sub> tasks.

The number of tasks of each kind in both worksheets have been summarised in Table 2.2 (see next page).

This worksheet can be considered as an interpretation of the *knowledge to be taught*, a step ahead toward the *knowledge actually taught*. This interpretation is more demanding in terms of the type of algebraic transformations that it requires compared to the limits that are usually included in the national examination.

### **Worksheet 2 (WS2)**

Worksheet 2 (see Appendix 2.5), produced by teachers of Francisco Manyanga high school in Maputo, is more detailed than WS1. It presents a theoretical part, which I will refer to in the analysis of the knowledge block, some tasks with its solutions as examples, and a list of tasks to be solved. Although the theoretical part gives some examples of tasks belonging to MO<sub>2</sub>, most of the tasks to be solved belong to MO<sub>1</sub>.

The number of occurrences of each kind of task in the practical section of WS2 is also presented in Table 2.2 (see next page), in parallel with the results of the classification of WS1 tasks.

As with the first worksheet, the emphasis of this worksheet about limits of functions is on algebraic tasks. Nevertheless these tasks are not as difficult as the tasks of Worksheet1. The difficulty of the algebraic transformations required to solve these tasks is comparable to the difficulty of the tasks that appear in the exams.

In both worksheets, there are no graphical tasks.

In the section called *Solved tasks* [Exercícios resolvidos], the technique to be used for each kind of task is exemplified. They are basically the techniques described in the analysis of the national examinations. For indeterminate forms such as  $(1^\infty)$ , the following formula is presented:  $\lim_{x \rightarrow a} f(x)^{g(x)} = e^{\lim_{x \rightarrow a} [f(x)-1] \cdot g(x)}$ , without any proof or comment.

Besides these tasks belonging to MO<sub>1</sub>, the solved tasks section also presents some tasks belonging to MO<sub>2</sub>. Curiously none of these tasks are present in the section of tasks to be solved by the students. It seems that solutions of these tasks are presented to students, but they are not required to solve any of them.

**Table 2-2 Tasks about limits in the two worksheets**

	KINDS OF TASKS OF MO <sub>1</sub>	WS1	WS2
<b>T<sub>1</sub></b>	<b>CALCULATE THE LIMIT OF A FUNCTION WHEN <math>x \rightarrow a</math></b>		
<b>T<sub>1A</sub></b>	Limit of a continuous function	2	6
<b>T<sub>1B</sub></b>	Indeterminate form of a rational function when $x \rightarrow a$	15	16
<b>T<sub>1C</sub></b>	Indeterminate form involving square roots	25	15
<b>T<sub>1D</sub></b>	Indeterminate form involving a trigonometric function	16	10
<b>T<sub>2</sub></b>	<b>CALCULATE THE LIMIT OF A FUNCTION WHEN <math>x \rightarrow \pm \infty</math></b>		
<b>T<sub>2A</sub></b>	Limit of a polynomial, a rational or an irrational function when $x \rightarrow \pm \infty$	30	27
<b>T<sub>2B</sub></b>	Indeterminate form such as $1^\infty$	7	15
<b>T<sub>3</sub></b>	<b>DETERMINE THE LIMIT OF A FUNCTION FROM THE GRAPH</b>	0	0
<b>T<sub>4</sub></b>	<b>STUDY THE CONTINUITY OF A FUNCTION</b>		
<b>T<sub>4A</sub></b>	Discussion of the continuity of a function using its analytical expression	7	2
<b>T<sub>4B</sub></b>	Discussion of the continuity of a function using its graph	0	0

### *Evaluation of the practical block*

The results of the analysis of the tasks in the national examinations and in the two worksheets from Maputo high schools are consistent with the analysis done some years ago. Students are mainly required to calculate limits using algebraic transformations. Some graphical tasks appear in the exams but the teachers do not seem to take account of these when designing the worksheets.

Let's have a closer look at this MO about limits of functions of Mozambican secondary school, using some criteria suggested by Chevallard (1999) to evaluate each component of a local or specific MO.

To evaluate the kinds of tasks, Chevallard proposes the following criteria (1999: 258):

- Identification criteria: Are the kinds of tasks well drawn and well identified?
- “Raison d’être” (rational) criteria: Is the study of these kinds of tasks motivated?
- Relevance criteria: Are these kinds of tasks relevant for actual and future mathematical activities of students.

In order to evaluate the corresponding techniques, he suggests answering some questions such as: Are the techniques effectively elaborated or only indicated? Are they easy to use? Do they have a satisfactory application? Are they understandable? Do they have some future? (1999: 259)

I will use these criteria to evaluate the MO’s practical block presented above.

#### *Evaluation of the kinds of tasks*

Identification criteria: The kinds of tasks are not well identified in the syllabus. As a consequence teachers must refer to national exams or worksheets developed by more experienced colleagues to identify the kinds of tasks to be taught. This interpretation of the syllabus made through the worksheets can differ from school to school. In fact the same kinds of tasks are present in the two worksheets analysed here, but the difficulty of their solution is quite different.

“Raison d’être” (rational) criteria: The study of these kinds of tasks does not seem to be motivated. Students do not need to understand the limit concept to solve these tasks. They only need to know algebraic transformations. Furthermore, there are no tasks which apply the result of a calculation, for example sketching the graph of a function or applying the limit concept to some word problem. As a consequence, students cannot understand the reason why they have to learn all these techniques to calculate limits. These tasks and techniques appear as an extension of algebraic work, without any connection with the limit concept.

Relevance criteria: The kinds of tasks solved by students in Mozambican secondary schools do not seem to be relevant for their actual and future mathematical activity. In fact, the main application of limits in Grade 12 is to define the derivative. In order to understand the definition of the derivative, students need to understand the limit concept as well as be able to use algebraic transformations to calculate limits.

Students who will enter university will also need limits in the calculus course, as a basic concept for other mathematical concepts such as integrals, sequences and series. In that case also, the understanding of this concept will be essential.

#### *Evaluation of the techniques*

Using Chevallard's criteria to evaluate the techniques, I would say that most of the techniques to solve algebraic tasks are not effectively elaborated, nor even indicated in the syllabus. They are indicated in the correction guides of national examinations, and in WS2, but not really elaborated. In WS2, there are some comments such as “to handle this indeterminate form, we factorise the highest power of the variable” [Para levantar esta indeterminação coloca-se em evidência a variável de maior grau]. There is no explanation of the reason why this technique would solve some specific indeterminate form. Furthermore, when they enter university, students learn L'Hôpital's Rule, which enable them to calculate most of these limits in a more simple way. It is important to teach these techniques using simple cases, because they will support results from L'Hôpital's Rule and can be used as alternative explanations. However they do not need to be taught in much detail because they are *weak techniques*. They will be substituted by L'Hôpital's Rule, which is a stronger technique.

As for the graphical tasks, there is no mention of the techniques to be used to solve them in the syllabus or in the correction guides of national examinations. The worksheets do not include any graphical work.

What kind of knowledge block can justify the kinds of tasks and techniques found in such a practical block? This is what I will analyse now.

### **2.3.2 The knowledge block (kinds of technology/theory)**

#### *The syllabus*

In the syllabus for Grade 12 (Ministério da Educação, 1997), the following objectives relate to the knowledge block of the chapter dedicated to limits of functions:

- |  |   |
|--|---|
| - explicar a noção de limite de uma função;  | - to explain the notion of limit of a function;   |
| - definir limite de uma função $f(x)$ quando $x \rightarrow a$ sendo $a$ um valor finito e quando $x \rightarrow \infty$ ; | - to define the limit of a function $f(x)$ when $x \rightarrow a$ , where $a$ is a finite value and when $x \rightarrow \infty$ ; |
| - explicar e aplicar as regras das operações com limites de funções;   | - to explain and apply the operating rules for limits of functions;   |
| - explicar e definir função contínua num ponto e função contínua num intervalo. (1997: 31)                                 | - to explain and to define continuous function at one point and in an interval.   |

In these objectives, the two levels of the ecological constraint to the existence of a MO inside an institution are present. On the one hand, it must be understandable. This is reflected by the objectives introduced by the verb “to explain”: students should be able to show that they understand, explaining the notion of limit of a function, the operation rules, and what a continuous function is. On the other hand, the existence of this MO must be justified, in reference to the scholarly mathematical knowledge. This demand of mathematical rigour is reflected by the objectives introduced by the verb “to define”: students should be able to define the limit of a function  $f(x)$  when  $x \rightarrow a$  and when  $x \rightarrow \infty$ , and to define a continuous function at one point and in an interval. We will see later that these two constraints can be in conflict.

The methodological orientations for Unit V are entirely devoted to the knowledge block (see Appendix 2.2). I will summarise them as follows:

- The teacher is expected to teach the formal definition, but students are not expected to memorise or use this definition;
- Special limits are not expected to be demonstrated, but an intuitive explanation, using the numerical or the graphical register, is to be done.

There is some contradiction in the knowledge block as presented in the objectives and methodological orientations of the syllabus. According to the objectives, students should be able to define the limit of a function when  $x \rightarrow a$  and when  $x \rightarrow \infty$ . According to the methodological orientations, they are not expected to know that definition. This contradiction appears in the new version of the syllabus. In fact the former syllabus (Ministério da Educação, 1993) reads:

A definição de limite de uma função (com a notação  $\varepsilon$ - $\delta$ ) costuma ser uma coisa completamente abstracta e incompreensível para os alunos. Assim sendo, o professor deve tentar primeiro dar uma ideia intuitiva deste conceito e a seguir tentar construir com base nesta ideia inicial, a definição propriamente dita. (1993: 15)

The definition of the limit of a function (with the  $\varepsilon$ - $\delta$  notation) is usually entirely abstract and impossible for students to understand. Therefore, the teacher must first try to give an intuitive idea of this concept and then attempt to construct the definition itself upon this initial idea.

In the 1997 syllabus, the first assertion has been removed. This modification is probably the consequence of discussions that we had in national seminars where, with my colleague, we presented the results of our previous research. On the one hand, teachers recognise that the definition is too complicated for students to understand; on the other hand they are very reluctant to teach a mathematical concept without giving a scientific definition. As a consequence, in the new version of the syllabus, it is not clear whether the formal definition should be taught or not.

### ***The national examinations***

The national examinations do not give any information about the knowledge block. The correction guides only present the solutions of the tasks without any justification. In fact, from national examinations, it could be inferred that students do not need to understand the technologies or theories inherent in the topic limits.

### ***The worksheets***

As said before, WS1 (see Appendix 2.4) only presents a list of algebraic tasks and their answers. There is no reference to the knowledge block about limits.

WS2 can be divided into three sections: a theoretical part (Section 1), solved examples with some technological comments (Section 2), and a list of tasks to be solved (Section 3). Section 3 (see Appendix 2.5) has already been analysed within the practical block.

The theoretical section (Section 1) begins with an introduction of the  $\varepsilon$ - $\delta$  definition through a graphical representation, using the function defined by  $f(x) = x + \frac{1}{2}$ . It

leads to the following statement of this definition:

$$(\forall \varepsilon > 0)(\exists \delta = \delta(\varepsilon)): (|x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon).$$

A solved task shows how to prove that  $\lim_{x \rightarrow 2} (3x + 1) = 7$  using the definition. It is followed by four tasks to use the same technique. They are clearly  $T_6$  tasks belonging to  $MO_2$ .

The following section presents definitions of one-sided limits, using sequences. For example, the left limit is defined as follows:

Diz-se que  $c$  é limite de  $f(x)$  à esquerda no ponto de abscissa  $a$ , se e só se (sse) a toda a sucessão de valores de  $x$  tendentes para  $a$ , com valores menores do que  $a$ , corresponde uma sucessão de valores de  $f(x)$  tendentes para  $c$ , e escreve-se:  $\lim_{x \rightarrow a-0} f(x) = c$ .

We say that  $c$  is the limit of  $f(x)$  from the left at the point of abscissa  $a$ , if and only if to each sequence of  $x$ -values approaching  $a$ , with values less than  $a$ , corresponds a sequence of  $f(x)$ -values approaching  $c$ , and we write:  $\lim_{x \rightarrow a-0} f(x) = c$ .

The right limit is defined in similar terms, and followed by the conclusion that the limit exists only if the one-sided limits are equal.

Then the following properties of limits are stated:

- The uniqueness theorem;
- The properties about sum or difference, product, quotient, power, and roots of convergent functions (see section 2.2).

There is no reference to infinite limits or limits at infinity in this section.

The following section (Section 2) of this worksheet is entitled “Solved tasks” [Exercícios resolvidos]. Only two of these tasks belong to  $MO_2$ . The first one is to calculate one-sided limits. This task could be considered as a  $T_{4A}$  task of  $MO_1$ , but the technique used for its solution (through sequences) makes it a  $T_7$  task of  $MO_2$ . The second task aims to show that the limit of some function when  $x \rightarrow a$  does not exist (one piece-wise function and one rational function). The same technique is used; therefore I also classified it as  $T_7$  task of  $MO_2$ .

The other tasks of this solved tasks section belong to  $MO_1$  ( $5T_{1A}$ ,  $3T_{1B}$ ,  $4T_{1C}$ ,  $3T_{1D}$ ,  $5T_{2A}$ , and  $2T_{2B}$ ). The following technological elements can be found in their solution:

- The solution of some tasks begin with the statement “The application of the theorems about limits leads to an indeterminate form such as  $\infty - \infty$ ” (or  $0 \cdot \infty$ )



[A aplicação dos teoremas sobre limites conduz a uma indeterminação do tipo  $\infty - \infty$  (ou  $0 \cdot \infty$ )].

- Some of the techniques are commented on. For example, before the calculation of  $\lim_{x \rightarrow 1} \frac{x^4 - 5x^2 + 4}{x^2 + x + 2}$ , it reads: “To calculate this indeterminate form, we factorise both terms of the fraction” [Para levantar esta indeterminação decompõem-se em factores ambos os termos da fracção]. Nevertheless there is no explanation of why this factorization is possible and why it could eventually solve the indeterminate form.

### ***Evaluation of the knowledge block***

To evaluate the technology, Chevallard suggests asking questions such as: Are statements justified or considered as evident, natural or already known? Are justifications adapted to their use? Are they explanatory? (1999:261)

To evaluate the corresponding theories, he suggests questions such as: Are elements of the theory explicit? Are they implicit? What can they highlight? What can they justify? (1999: 262)

Regarding the syllabus, my analysis shows that the knowledge block is not well defined. Looking at this syllabus, teachers would not be able to know which definition of limit they are expected to teach, what theorems they are expected to justify or to prove. There is no reference to the theory underlying the study of limits of functions.

The worksheets, which are an interpretation of the syllabus, reflect this ambiguity. WS1 omitted the knowledge block. The teachers who elaborated WS2 tried to fill this knowledge block with some definitions and proofs. Nevertheless, there are some discrepancies between the practical block and the knowledge block of this worksheet that I will highlight now.

The first definition, given at the beginning with an example and two tasks, is never applied afterwards. The second definition through sequences, known in Mozambique as Heine’s definition, is given in the theoretical section of the worksheet without any

example, applied in Section 2 (6 tasks), but students are not required to apply it any more in section 3.

The properties of limits stated in section 1 only relate to limits when  $x \rightarrow a$ , where  $a$  is a finite value. Nevertheless, in Section 2, some examples are given of limits when  $x \rightarrow \infty$ . These kinds of tasks are also present in section 3. I put side by side the three sections of this worksheet in Table 2.3. As for  $MO_1$ , I subdivided some  $MO_2$  tasks as follows.

**T<sub>6A</sub>**: Prove the existence (or non-existence) of the limit of a function  $f$  as  $x \rightarrow a$ , where  $a$  is a real number.

**T<sub>6B</sub>**: Prove the existence (or non-existence) of the limit of a function  $f$  as  $x \rightarrow \pm\infty$ .

**T<sub>7</sub>**: Prove the existence (or non-existence) of one-sided limits for certain kinds of functions (such as monotonic functions).

**T<sub>8A</sub>**: Prove the properties used to justify the way certain limits of functions are calculated when  $x \rightarrow a$ , where  $a$  is a real number.

**T<sub>8B</sub>**: Prove the properties used to justify the way certain limits of functions are calculated when  $x \rightarrow \pm\infty$ .

All kinds of tasks of the reference mathematical organisation ( $MO_1$  and  $MO_2$ ) have been compiled in Appendix 2.6.

**Table 2-3 Comparison of the content of worksheet 2's three sections**

Reference mathematical organisation	Section 1 Theoretical approach	Section 2 Solved examples	Section 3 Students' tasks
$MO_2$	$\varepsilon$ - $\delta$ definition, 1T <sub>6A</sub> example ( $MO_2$ ), 4T <sub>6A</sub> tasks ( $MO_2$ )		
$MO_2$	Heine's definition (through sequences) ( $MO_2$ )	6T <sub>6A</sub> tasks ( $MO_2$ )	
$MO_1$ , T <sub>1</sub>	Properties of limits when $x \rightarrow a$ where $a$ is a finite value ( $MO_2$ )	5T <sub>1A</sub> , 3T <sub>1B</sub> , 4T <sub>1C</sub> , 3T <sub>1D</sub> ( $MO_1$ )	6T <sub>1A</sub> , 16T <sub>1B</sub> , 15T <sub>1C</sub> , 10T <sub>1D</sub> , ( $MO_1$ )
$MO_1$ , T <sub>2</sub>		5T <sub>2A</sub> , 2T <sub>2B</sub> ( $MO_1$ )	27T <sub>2A</sub> , 15T <sub>2B</sub> ( $MO_1$ )
$MO_1$ , T <sub>4</sub>			2T <sub>4A</sub> ( $MO_1$ )

This table clearly shows the disequilibrium between the content of the three sections of this worksheet.

With regard to  $MO_2$ , the knowledge block presented in the theoretical approach (the two definitions) does not correspond to any practical block when it comes to students' tasks. This can be seen as a result of the contradiction already mentioned between two constraints of the didactical transposition. On the one hand, the content must be justified, to ensure the reference to the scholarly knowledge, hence the theoretical part. On the other hand, it must be understandable, but secondary school students are not able to understand the theoretical part. The problem has been solved in this worksheet in a cosmetic way: the definitions are there, but students are not required to be able to use them.

On the contrary, some parts of the practical block of  $MO_1$  are not justified by any technological discourse. This is the case of the kinds of tasks  $T_2$  (limits at infinity) and  $T_4$  (continuity of functions). The definition of infinite limits or limits at infinity are absent, as well as the properties of limits when  $x \rightarrow \pm\infty$ . Nevertheless these properties are referred to in Section 2 by the sentence “the application of the theorems about limits leads to an indeterminate form ...” as if they were a natural extension of the same properties when  $x \rightarrow a$ . The properties of limits when  $x \rightarrow a$ , which refers to  $T_1$  tasks, are presented without any proof in the theoretical part of the worksheet, and referred to in the solved examples. No reference is done to continuity, even if students are asked to determine one-sided limits of piece-wise functions.

To summarise, this analysis shows that some elements of the technology can be found in the syllabus and in WS2. In the syllabus, traces of a technological discourse are present, but it is not very clear which definition of limits must be taught, and whether the rules about limits should be demonstrated or not. In WS2, we saw that the technological discourse, belonging to  $MO_2$  does not have any practical block, while the  $MO_1$  practical block is not justified by any systematic knowledge block.

Barbé et al. (2005) point out a similar situation in the Spanish secondary school curriculum: “the considered mathematical knowledge to be taught is composed of the disjoint union of the trace of  $MO_1$  and  $MO_2$ ” (2005: 245). In the same work,

besides their study of the *knowledge to be taught*, they also analyse the *knowledge actually taught* in two Spanish secondary school classes. They show that it was centred around two didactic moments: “the technological-theoretical moment” and the “moment of technical work, in which students ‘applied’ and ‘practised’ the techniques the teacher had just showed them through some typical examples” (2005: 261). They identify two didactic restrictions that affect these teachers’ practices.

- (1) *Specific didactic restrictions* arising from the precise nature of the knowledge to be taught. [...]
- (2) *Generic didactic restrictions* the mathematics teachers encounter when facing the problem of how to teach any proposed mathematical topic in a school institution. (2005: 239)

They argue that

The conjunction of the two kinds of restrictions determines to a large extent the knowledge related to limits of functions that can actually be taught in the classroom. (2005: 240)

I suggest that these two restrictions are also relevant to Mozambican teachers’ practice. In fact, considering the nature of the limit concept, it seems difficult for a teacher to teach  $\text{MO}_2$  at secondary school level. The  $\varepsilon$ - $\delta$  definition is very abstract, and it is necessary to have a good concept image of limits in order to understand it. What could be done at secondary school level is to help students develop their concept image of limits, in order to be able to understand the definition later, possibly at university. I later argue that this could be done using, on the one hand, the numerical and the graphical register and, on the other hand, word problems taken from other areas, such as geometry, biology or economics, which would show students possible applications of this concept.

First of all, this means that the syllabus should be revised. What kind of didactical transposition can a teacher do with the knowledge to be taught found in the syllabus and national examinations? There is little room for innovation. It clearly appears in the examinations that students should be able to calculate indeterminate forms through algebraic transformations. Teachers who look back at the scholarly mathematical knowledge would probably add some technological elements, as the definition and some properties of limits, as stated in the syllabus. This is the case of

the teachers who elaborated WS2. Others would not worry in teaching something that they know students will not understand and will never use in practice, and go directly to the calculations, as in WS1.

In the case of Mozambique, I would like to add a third restriction, which is teachers' mathematical knowledge about limits of functions. My argument is that, in order to help students deepen their concept image on limits, teachers need to develop their own knowledge of this concept. If teachers' personal relation to limits fits the institutional secondary school relation to this concept, if they are “good subjects” of the institution, to use Chevallards' terms, they will not be able to teach limits in a more elaborate way than teaching calculations. To better understand the possible personal relation to limits of functions of a secondary school teacher trained in Mozambique, I needed to look at Pedagogical University relation to this concept.

## 2.4 The Pedagogical University's relation to limits of functions

At the Pedagogical University (PU), the topic “Limits” is part of the Calculus I course [Análise Matemática I] taught during the first semester. In the syllabus, it reads:

Sucessões numéricas e subsucessões; noções de limite duma sucessão numérica e de uma função; unicidade do limite; operações algébricas sobre os limites (limites da soma, produto e quociente); limite de uma função composta; passagem ao limite em desigualdades; limites laterais; continuidade do conjunto  $\mathbb{R}$ ; limite de uma sucessão numérica monótona; números decimais infinitos; limites da razão entre o seno e o seu argumento quando este tende para zero; o número “e”; teorema de Bolzano-Weierstrass.

Numerical sequences and subsequences; notion of limit of a numerical sequence and of a function; uniqueness of the limit; algebraic operations with limits (limits of a sum, a product, a quotient); limit of a compound function; limits in inequalities; one-sided limits; continuity of  $\mathbb{R}$ ; limit of a monotonous numerical sequence; infinite decimal numbers; limits of the ratio between sinus and argument when this later tends to zero; the number “e”; Bolzano-Weierstrass theorem.

No further indications are given on how to teach this topic.

Furthermore, there is no official textbook written in Mozambique. For this reason it is rather difficult to analyse the knowledge to be taught. Nevertheless, two Russian textbooks, translated into Portuguese, are often used at university level: one by Demidovitch (1977), and the other one by Piscounov (1977). We will see later that in fact all teachers involved in the research group referred to these textbooks as the

books that they used in their calculus courses at university level. In order to have a view of the taught knowledge about limits of function at PU, I analyse in this section the chapter about limits of functions in these two textbooks as well as the exercise-book of one of the teachers involved in the research group.

### 2.4.1 Practical block (kinds of tasks/techniques)

#### *The textbook by Demidovitch*

The textbook by Demidovitch was first published in Portuguese in 1977 and has been republished several times since. The part dedicated to limits (1977: 23-33) begins with a theoretical introduction, which I will refer to in the knowledge block. The tasks that follow are to prove the existence of the limit of sequences (n° 166-67) or a function (n° 168). I classified them as  $T_{6B}$  tasks for the first two ones and  $T_{6A}$  for the last one. Then students are required to explain three limits (n° 169), such as  $\lim_{x \rightarrow +0} \log x = -\infty$ . These tasks seem to be an application of the formal definition. Then they would be  $MO_2$  tasks,  $T_{6A}$  for the first one and  $T_{6B}$  for the other two. Task 170 asks to find the limit of four sequences. I have classified it as  $T_{2A}$ , even if the technique to be used to solve these tasks is not explicit. Tasks 171 to 180 are to determine the limit of sequences when  $n \rightarrow \infty$ . As for task 170, it is not clear which techniques should be used to perform this kind of task. I have classified them as  $T_{2A}$  tasks. Then some rules are given to calculate limits of functions, each one followed by at least an example and several tasks with increasing difficulties (tasks 181 to 263). For example, it reads.

Outro método, através do qual pode-se encontrar o limite, a partir de uma expressão irracional, é o transporte da parte irracional do numerador para o denominador, ou ao contrário, do denominador para o numerador (1977: 27)

Another method, through which we can find the limit of an irrational expression, is the shifting of the irrational part from the numerator to the denominator, or at the opposite, from the denominator to the numerator.

This statement is illustrated by the following example

$$\lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}} \quad (a > 0).$$

Then there are 13 tasks to apply this technique (n° 203 to 215), the first one being

$$\lim_{x \rightarrow 7} \frac{2 - \sqrt{x-3}}{x^2 - 49} \text{ and the last one } \lim_{x \rightarrow \infty} \left( x + \sqrt[3]{1-x^3} \right).$$

There is no justification for this technique.

These tasks have been classified either as T<sub>1B</sub>, T<sub>1C</sub>, T<sub>1D</sub>, T<sub>2A</sub> or T<sub>2B</sub> depending on the kind of function involved (see Appendix 2.7).

In page 29 of this textbook, there is an explanation on how to calculate limits such

as  $\lim_{x \rightarrow a} \varphi(x)^{\psi(x)}$  by using the formula  $e^{\lim_{x \rightarrow a} [\varphi(x)-1]\psi(x)}$ , again without any proof. For

the other techniques a student with good understanding of algebraic transformations is able to understand when and why each technique works. But in the case of the formula  $e^{\lim_{x \rightarrow a} [\varphi(x)-1]\psi(x)}$ , it is very difficult to find out the reason why the two limits are equivalent. The student is just required to memorise and apply the formula.

Tasks 264 to 270 are to calculate one-sided limits, but without any reference to continuity. They have been classified as 3 T<sub>1B</sub>, 1 T<sub>1D</sub> and 3 T<sub>2A</sub> tasks (Appendix 2.7).

Tasks 271 to 287 are new tasks in relation to the reference MO described in this chapter. They are tasks where the limit concept appears as a tool to solve some other problem. These applications belong to the following areas:

- Graph sketching (n° 271-275). For example, task 275 asks to sketch the graph of the function  $y = \lim_{n \rightarrow \infty} \sqrt[n]{1+x^n}$ . The answer to this question presented at the end of the book indicates:  $y=1$  when  $0 \leq x \leq 1$ ;  $y=x$  when  $1 < x < \infty$ .
- Arithmetic tasks. For example, task 276.

Transformar em fracção ordinária a fracção periódica mista dada  $\alpha = 0,13555\dots$ , considerando-a como limite da fracção finita correspondente. (Demidovitch, 1977: 32)

Write the periodic mixed fraction [meaning recurring decimal]  $\alpha = 0,13555\dots$  as an ordinary fraction, by considering it as the limit of the corresponding finite fraction.

- Geometric tasks. Example, task 278.

Achar o limite do ângulo interno de um

Find the limit of the internal angle of a

polígono regular de  $n$  lados, quando  $n \rightarrow \infty$ . (p. 32)      regular  $n$ -sided polygon when  $n \rightarrow \infty$ .

- Other areas. For example, task 287 is an application of limits to chemistry.

I considered all these tasks as belonging to a new  $MO_1$  kind of task, coded as  $T_5$ .

**$T_5$ :** Apply limits in mathematics or in other disciplines

This section about limits is followed by a section entitled “Infinitesimals and infinites”. The first tasks of this section are to prove that some functions are infinitesimal when  $x \rightarrow \infty$  (n° 288, 289) or infinitely great when  $x \rightarrow a$  (n° 290). These tasks belong to  $MO_2$ , the first two being  $T_{6A}$  tasks and the third a  $T_{6B}$ . The following tasks aim to determine the infinitesimal degree of some functions. Some of them are linked to applications in geometry (n° 291, 292, 294, 295), and I have classified them as  $T_5$  of  $MO_1$ . In others, a function is given by its analytical expression. They belong to several kinds of  $MO_1$  tasks and have been classified as 5  $T_{1B}$ , 4  $T_{1C}$ , 5  $T_{1D}$  and 4  $T_{2A}$  (see Appendix 2.7).

The last section of this chapter is about continuity of functions. The first tasks of this section (n° 304-312) aim to prove the continuity of some functions. I have classified them as  $T_7$  of  $MO_2$ . Then there are some  $T_{4A}$  tasks of  $MO_1$  (n° 313-335).

The number of tasks of each kind, classified according to the type of task of the reference  $MO$  (see Appendix 2.6), is presented in the third column of Table 2.4 (DM, page 45).

The disequilibrium between  $MO_1$  and  $MO_2$  clearly appears in this table. In fact 90% of the tasks (167 out of 185) belong to  $MO_1$  while only 10% (167 out of 185) belong to  $MO_2$ .

### ***The textbook by Piscounov***

As with the Demidovitch textbook, the textbook by Piscounov has been printed several times since its first Portuguese edition in 1977. The chapter on limits and continuity of functions in this textbook is divided into a theoretical part (1977: 34-68) and a list of tasks with their answers (1977: 69-72). The classification of these tasks can be found in the fourth column of Table 2.4 (PSC, page 45). All of them belong to  $MO_1$ . Most of the techniques to be used to solve these tasks are illustrated



in the theoretical part. They are:  $T_{2A}$  (example 1, p. 48),  $T_{1A}$  (examples 2 and 3, p. 49-50),  $T_{1B}$  (example 4 and 5, p. 50),  $T_{1D}$  (4 examples, p. 53-54),  $T_{1D}$  (4 examples, p. 53-54),  $T_{2C}$  (4 examples, p. 58-59), and  $T_{4A}$  (examples 1-10, p. 61-64). The tasks have been classified using the same criteria as for Demidovitch textbook tasks (see Appendix 2.7). They are presented in the fourth column of Table 2.4 (PSC, p. 46).

### ***The exercise-book***

I looked at the calculus course notes from one of the teachers involved in my research and analysed the part dedicated to limits. This section presents theoretical parts, given during lectures, and practical parts, corresponding to seminars. In this section, I will refer to the practical parts. The first one includes five examples and 51 tasks to calculate limits of functions, all of them coming from the textbook by Demidovitch and belonging to  $MO_1$ .

The examples, solved by the lecturer, introduce the technique to be used to solve some of the tasks. They belong to  $T_{2A}$  (task 185 from Demidovitch),  $T_{1B}$  (task 193),  $T_{1C}$  (tasks 199 and 200), and  $T_{2C}$  (example 9, p. 29-30 of the same book). The tasks solved by the student are the remaining from task 181 to task 262 of the same textbook, excluding task 228.

The second practical part is dedicated to continuity of functions, and includes 15  $T_{4A}$  tasks, all of them from Demidovitch. The classification of all these tasks is shown in the last column of Table 2.4 (ExB, p.46).

### ***Evaluation of the practical block***

The practical block of the MO about limits of functions at the Pedagogical University is not very different from the one in secondary schools. Most of the tasks to be solved by students in the textbooks belong to  $MO_1$ . Some of the tasks from Demidovitch textbook belong to  $MO_2$ , but they were not used by the lecturer, according to the exercise-book that I analysed. A kind of task which does not appear in the school context is present in both textbooks, which is applying the limit concept as a tool to solve some mathematical or other sciences' problems. This kind of task could be useful to give some meaning to the limit concept. Applying this concept in some word problem could be a justification for

students to learn this topic. Nevertheless, these tasks were also left apart by the lecturer when selecting tasks from Demidovitch for the seminars. As a consequence, most of the tasks solved by the students-teachers were algebraic tasks.

**Table 2-4 Number of tasks about limits in two textbooks and an UP exercise-book**

	KINDS OF TASKS	DM	PSC	ExB
	<b>TASKS OF MO<sub>1</sub></b>	<b>167</b>	<b>60</b>	<b>66</b>
<b>T<sub>1</sub></b>	<b>CALCULATE THE LIMIT OF A FUNCTION WHEN <math>x \rightarrow a</math></b>			
<b>T<sub>1A</sub></b>	Limit of a continuous function		3	
<b>T<sub>1B</sub></b>	Indeterminate form of a rational function when $x \rightarrow a$	16	10	8
<b>T<sub>1C</sub></b>	Indeterminate form involving square roots	16	7	10
<b>T<sub>1D</sub></b>	Indeterminate form involving a trigonometric function	31	17	15
<b>T<sub>2</sub></b>	<b>CALCULATE THE LIMIT OF A FUNCTION WHEN <math>x \rightarrow \pm \infty</math></b>			
<b>T<sub>2A</sub></b>	Limit of a polynomial, a rational or an irrational function when $x \rightarrow \pm \infty$	37	12	14
<b>T<sub>2B</sub></b>	Indeterminate form such as $1^\infty$	23	8	4
<b>T<sub>3</sub></b>	<b>DETERMINE THE LIMIT OF A FUNCTION FROM THE GRAPH</b>			
<b>T<sub>4</sub></b>	<b>STUDY THE CONTINUITY OF A FUNCTION</b>			
<b>T<sub>4A</sub></b>	Discussion of the continuity of a function using its analytical expression	23	3	15
<b>T<sub>4B</sub></b>	Discussion of the continuity of a function using its graph			
<b>T<sub>5</sub></b>	<b>APPLY LIMITS IN MATHEMATICS OR IN OTHER DISCIPLINE</b>	21		
	<b>TASKS OF MO<sub>2</sub></b>	<b>18</b>	<b>0</b>	<b>0</b>
<b>T<sub>6A</sub></b>	Prove the existence (or non-existence) of the limit of a function $f$ as $x \rightarrow a$ , where $a$ is a real number	4		
<b>T<sub>6B</sub></b>	Prove the existence (or non-existence) of the limit of a function $f$ as $x \rightarrow \pm \infty$	5		
<b>T<sub>7</sub></b>	Prove the existence (or non-existence) of one-sided limits for certain kinds of functions (such as monotonic functions)	9		
<b>T<sub>8A</sub></b>	Prove the properties used to justify the way certain limits of functions are calculated when $x \rightarrow a$ , where $a$ is a real number			
<b>T<sub>8B</sub></b>	Prove the properties used to justify the way certain limits of functions are calculated when $x \rightarrow \pm \infty$			

The situation is identical to the one described in the evaluation of the practical block of the MO about limits in Mozambican secondary schools. The only difference is that the algebraic transformations involved are more complicated.

Considering that this was an academic teacher training course, it was to be expected that the knowledge block should be more developed than at secondary school level. I will now analyse this knowledge block using the same sources.

### 2.4.2 Knowledge block (kinds of technology/theory)

#### *The textbook by Demidovitch*

The section entitled “Limits” begins with the following definition of limit of a sequence.

O número $a$ denomina-se <i>limite</i> da sucessão $x_1, x_2, \dots, x_n, \dots$ : $\lim_{n \rightarrow \infty} x_n = a$ , se para qualquer $\varepsilon > 0$ existe um número $N=N(\varepsilon)$ tal que $ x_n - a  < \varepsilon$ sendo $n > N$ . (1977: 23)	The number $a$ is called <i>limit</i> of the sequence $x_1, x_2, \dots, x_n, \dots$ : $\lim_{n \rightarrow \infty} x_n = a$ , if for any $\varepsilon > 0$ there exists a number $N=N(\varepsilon)$ such that $ x_n - a  < \varepsilon$ where $n > N$ .
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An example is given on how to use this definition to demonstrate

that  $\lim_{n \rightarrow \infty} \frac{2n+1}{n+1} = 2$ . Then it defines limit of a function when  $x \rightarrow a$  ( $\varepsilon$ - $\delta$

definition) and when  $x \rightarrow \infty$ . After defining the lateral limits, an example is

presented of one-sided limits of the function  $f(x) = \arctan \frac{1}{x}$  when  $x$  goes to zero.

This constitutes the theoretical part of this section about limits. As said before in the analysis of the practical block, tasks 181 to 270 are to calculate different kinds of limits, using different techniques. Before each kind of task, there is a short explanation of the technique, without any justification. For example:

Para o cálculo dos limites abaixo relacionados, é útil saber que se existe e é positivo o $\lim_{x \rightarrow a} f(x)$ , então $\lim_{x \rightarrow a} [\ln f(x)] = \ln \left[ \lim_{x \rightarrow a} f(x) \right]$ . (1977: 30)	To calculate the limits above, you must know that if $\lim_{x \rightarrow a} f(x)$ exists and is positive, then $\lim_{x \rightarrow a} [\ln f(x)] = \ln \left[ \lim_{x \rightarrow a} f(x) \right]$ .
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None of these explanations is demonstrated, nor even justified.

This textbook presents the same dichotomy between the MO<sub>2</sub> knowledge block, which corresponds to only two tasks for students (168,169, p. 25), and the MO<sub>1</sub> practical block, without any theoretical justification.

### ***The textbook by Piscounov***

As already stated, the chapter on limits in the textbook by Piscounov (1977) includes an extended theoretical part. It begins with an explanation of what  $x \rightarrow a$  means,

using the  $\varepsilon$ -notation, using the sequences  $x_n = 1 + \frac{1}{n}$  and  $x_n = 1 + \frac{1}{2^n}$  as examples,

and showing the uniqueness of the limit. Then the limit of a function when  $x \rightarrow a$  is defined in terms of  $\varepsilon$ - $\delta$ , with a graphical interpretation and examples on how to demonstrate some limits using this definition (1977: 39-40, examples 1 and 2). The definition is then extended to the case  $x \rightarrow \infty$ , again with a graphical interpretation and an example (1977: 40, example 3). For infinite limits, the definition is also exemplified and illustrated (1977: 41-42, examples 1 and 2), as well as for limited functions (1977: 42-43, examples 3 and 4). After a section dedicated to “infinitely small quantities and its fundamental properties” (1977: 45-48), the following theorems about limits are stated and demonstrated:

- Limits of sum, product, quotient of variables, demonstrated using infinitely small quantities;
- The squeeze theorem.

This chapter follows with the special limit  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , illustrated by a graph and

demonstrated using the squeeze theorem. The number  $e$  is then introduced through

the sequence  $\left(1 + \frac{1}{n}\right)^n$  and the squeeze theorem used to show that

$2 < \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n < 3$ . This leads to the special limit  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ . After an

introduction of natural logarithms, a section is dedicated to the properties of continuous functions. The theoretical part of this chapter ends with a section entitled “Comparison of infinitely small quantities”.

### ***The exercise-book***

The theoretical part of the exercise-book that I have analysed is similar to the knowledge block of Piscounov textbook, but sometimes the order of the sections is different. A comparison of the contents of these two knowledge blocks can be seen in Table 2.5 (see next page).

From this table we can deduce that the book by Piscounov was the main source used by the lecturer to prepare his lectures: most of the definitions are exactly the same; the demonstrations come from that book, sometimes with some further comment; the examples given also come from that book. The first part of the chapter (limit of a function) is rather developed, and very similar to the book. The next sections do not enter into so many details, as if the lecturer was running out of time.

When it comes to infinitesimals, the order of the book has been reversed. While in the book infinitesimals are introduced in Section 4, and their comparison made in Section 11, in the exercise-book the two sections about this topic were put together just before the study of continuity. The reason why infinitesimal has been separated in two sections in the book seems to be that infinitesimals are used in Section 5 to prove theorems about limits. One of these demonstrations also appears in the exercise-book, using the concept of infinitesimal which has not been introduced yet.

The section dedicated to continuity of functions in the exercise-book is quite different from Piscounov textbook. The lecturer probably used another source to prepare this section.

### ***Evaluation of the knowledge block***

The analysis of the two textbooks and the exercise-book clearly shows that the book by Piscounov was the reference for the knowledge block of the *knowledge to be taught* for the PU lecturer.

The knowledge block in Piscounov textbook is well structured, all statements are justified, many of them illustrated with examples and sometimes with graphs.

**Table 2-5 Comparison of the knowledge block in Piscounov textbook and in the exercise-book**

Piscounov	Exercise-book
1. Limit of a variable quantity ( $\varepsilon$ -definition, 2 examples) Infinitely great variable quantity (definition, 1 example)	
2. Limit of a function (when $x \rightarrow a$ , $\varepsilon$ - $\delta$ definition, 2 examples; when $x \rightarrow \infty$ , $\varepsilon$ - $\delta$ definition, 2 examples)	Limit of a function (same definitions, same examples)
3. Functions which tend to infinity (definition, 4 examples) Limited functions (3 definition, 2 theorems)	Functions which tend to infinity (same definition, without the examples) Limited functions (same definition, without the example)
4. Infinitesimals and its fundamental properties (definition, 4 theorems with proofs)	
5. Fundamental theorems on limits (seven theorems with a demonstration and examples)	Main properties of limits (four from seven theorems, demonstration of the first one, no examples)
6. Limit of the function $\frac{\sin x}{x}$ when $x \rightarrow 0$ (demonstration, 4 examples)	Statement $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ without demonstration, no example
7. The number $e$ (introduction of $e$ , demonstration of the special limit $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ , 4 examples)	Same demonstration of the special limit $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ , no example
	Infinitesimals and its fundamental properties (From section 4: same definition, 2 of the theorems without proof; from section 11: same definitions, same examples, one of the theorem with proof), list of equivalent infinitesimals
8. Natural logarithms	
9. Continuity of functions	Functions continuous at a point (definition, properties, classification of discontinuity points; different from section 9 and 10)
10. Properties of continuous functions	
11. Comparison of infinitesimals (4 definitions with examples, 2 theorems with proofs)	

The didactical transposition made by the lecturer is not so systematic:

- Some of the theorems are not demonstrated, as if they were considered as natural;
- Most of the 25 graphs that illustrated this chapter (Figures 29 to 53) do not appear in the exercise-book (only Figure 29 is reproduced in this exercise-book);
- The order of some sections has been changed and, as a consequence, the concept of infinitesimal is used before being defined.

Obviously, the student-teacher's exercise-book cannot be considered as corresponding exactly to the *knowledge actually taught*, but only as an indication of some of this knowledge. Even if this textbook is well organised, it is possible that this student-teacher did not take notes of everything done in the classroom. Students often only take note of what the teacher writes on the blackboard, and not of what s/he says. The teacher's comments and explanations often complement the written content. For example, in this case, the lecturer could have explained what an infinitesimal is when proving the theorem about the limit of a sum of functions. As for the graph, other graphs can have been presented by the lecturer, with no time for the student-teacher to reproduce them in his exercise-book. They can also have been referred to in the textbook.

Nevertheless, even considering the limitations above indicated of this analysis of the *knowledge actually taught* at PU, I would say that the statements made in the exercise-book are not as justified and explanatory as they appear in the textbook by Piscounov. We can presume that, as stated by the teachers involved in the project during the interviews, constraints of time and students' understanding probably limited the lecturer when giving his classes about limits of functions.

However, looking at the knowledge block as seen through the exercise-book, I would ask some questions about the results of this didactical transposition for the student-teachers' learning, as for example:

- What conception of the role of definition in mathematics can be formed through this limit chapter? Formal definitions are given but never applied in practice. These student-teachers could conclude that defining a concept is only

a formal activity. They could not see the definition as a reference for the concept.

- What conception of the role of proofs in mathematics can be developed by the student-teachers? Many theorems are stated but not proved. Student-teachers could deduce that proof is not important and that, as teachers, they can teach their own students rules without any justification.

Furthermore, the study of the knowledge about limits of functions as taught at PU highlighted the same dichotomy between the practical block and the knowledge block as at secondary school. As expected at university level, the knowledge block is more developed and structured than the knowledge to be taught at secondary school, closer to the scholarly mathematical knowledge, but not really rigorous. Nevertheless it remains completely separated from the practical block, as if there were no link between them.

What conception of limits would a teacher trained through this institution develop?

Considering the weak development of the concept image of limits in secondary schools, can we expect that teachers trained at PU will form a better image of limits? Or will they reinforce the idea that limits are only about calculations? How will they teach limits in secondary schools? I will try now to give some answers to these questions.

## 2.5 Conclusion

In this chapter, I have analysed the Mozambican secondary school and the Pedagogical University relation to limits of functions, using the notion of mathematical organisation (MO) as developed by Chevallard (1999) and the reference mathematical organisation as defined by Barbé et al. (2005). This reference mathematical organisation has been structured into two regional MOs:  $MO_1$ , the algebra of limits, and  $MO_2$ , the existence of limits.

Concerning the Mozambican secondary school's relation to limits, the analysis of the *knowledge to be taught*, through the syllabus, the national examinations and two worksheets, shows a dichotomy between these two regional MOs. Some trace



of  $MO_2$  can be found in the syllabus and in one of the worksheets, but is completely absent in the examinations and in the other worksheet. Most of the tasks related to limits are algebraic, without any technological justification for the use of the required techniques. As a consequence, we can surmise that students concluding secondary school in Mozambique would have a poor concept image of limits. This was also the conclusion drawn by Mutemba (2001) in her analysis of Mozambican students' understanding of this concept. She concluded that "the majority of the students mostly conceived of the limit as a static number, without any relation with the limiting process" (2001: 101). She pointed out the "dominance of the *procedural image* that guided students to an algebraic representation and their trust in algebraic solution" (2001: 96).

The nature of the limit concept, which is a very abstract concept, and the difficulties of understanding  $MO_2$  for secondary school students constrain the possible didactical transposition at secondary school level, as pointed out by Barbé et al. (2005). It does not seem possible to extend the teaching of limits of functions at  $MO_2$  at this level. Nevertheless, I argue that students' concept image of limits of functions could be more developed at secondary school, by a more elaborated teaching of  $MO_1$ . In my opinion, this could be done in two ways.

On the one hand, students' work in different registers, particularly in the numerical and graphical registers, could be developed. The numerical register is sometimes used in Mozambique during the introduction of limit of a sequence in schools, but is hardly used in tasks to be performed by students. As a consequence, students are not able to use numerical values to intuitively find out some limit. When they enter university, some students are able to calculate

complicated limits, but do not see intuitively that  $\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$ . Solving some

numerical tasks could help them develop a better concept image of limits, understand them intuitively, and draw by themselves some conclusions about the way limits relate to one another. In the same way, in Mozambique the work in the graphical register is also often limited to the introduction of limits. For example, the only graph from Piscounov textbook found in the exercise-book is the first one, which is an illustration of the  $\varepsilon$ - $\delta$  definition. As with the numerical

representation, this graphical interpretation is done by the teacher, but no further graphical task is given to students. Tasks to read limits from a graph hardly occur in the national examinations, and do not appear in the worksheets for secondary schools, or in the PU exercise-book. The limits of a function are never used to sketch its graph. These two kinds of tasks (reading limits from a graph, sketching a graph using the limits of the function) could also help students develop a better concept image of limits.

On the other hand, tasks could be introduced to apply limits in mathematics or in other sciences. We already saw that some word problems were present in the textbooks by Demidovitch and Piscounov. Other small tasks can be found in textbooks from other countries, to apply limits in geometry, biology, economics, and physics. This kind of task could show students the usefulness of the limit concept.

In order to extend the teaching of  $MO_1$  in schools, instead of using only algebraic tasks, Mozambican teachers should be made aware of the limitations of the present didactical transposition, and of the possibilities of its extension. Does the way limits of functions are taught at Pedagogical University enable them to take this step aside and look critically at the *knowledge to be taught* in schools?

The analysis of the *knowledge actually taught* at Pedagogical University, through two textbooks and a student's exercise-book, presents a knowledge block much more elaborated than at secondary school level. However, the same dichotomy exists between the merely algebraic practical block belonging to  $MO_1$  and the knowledge block belonging to  $MO_2$ , which is never used in practice. It is to be expected that the teachers trained at PU would consider the first step of the didactical transposition made by the secondary school institution as natural, and would not question the *knowledge* about limits of functions *to be taught* in schools. Even if they feel uncomfortable with this dichotomy between the teaching of the definition and the algebraic tasks, they would probably find a cosmetic solution, as the one done in the worksheets.

This leads me to the following question: What kind of knowledge does a mathematics teacher need in order to consciously perform the second step of the didactical transposition in the case of the limit concept?

This is the topic of the next chapter.

# **CHAPTER 3**

## **MATHEMATICS FOR TEACHING: DEVELOPING A GENERAL FRAMEWORK**

### **3 Mathematics for Teaching: Developing a General Framework**

In Chapter 2, I analysed the relation of Mozambican secondary schools and the Pedagogical University to the limit concept. I argued that the secondary schools' relation to this concept could be more elaborated, for example by using several registers, such as the numerical and the graphical registers, and by applying the limit concept in word problems from several areas. Nevertheless, Mozambican mathematics teachers' personal relation to limits, shaped by the institutions where they have met this concept, could be an obstacle to this elaboration, as argued in the previous chapter. To be able to influence the secondary schools' institutional relation to limits, teachers should be aware of the limitations of the present didactical transposition, both in terms of knowledge to be taught and of knowledge usually taught in classrooms. They should be able to provide a second step of this transposition (knowledge to be taught  $\rightarrow$  knowledge usually taught) different from the standard one. This means that their personal relation to limits should change, and this is only possible if they are in contact with the limit concept through a new institution, where this concept lives in a different way. This leads me to ask the following questions:

- (1) What knowledge does a Mozambican mathematics teacher need to know in order to teach limits of functions in schools in a more elaborated way?
- (2) What kind of institution could help teachers develop their personal relation to limits?

Question (1) derives from a more generic one:

- (3) What knowledge does a mathematics teacher need in order to teach a specific topic in schools?

Questions (1) and (2) will be the topics of Chapters 4 and 5 respectively. The present chapter addresses question (3).

In this chapter, I initially review the main ideas that have been developed about what kind of knowledge a mathematics teacher needs, in particular the link

between mathematical and pedagogical knowledge, and between theoretical and practical knowledge.

Then I present and discuss the framework developed by Even (1990, 1993) to analyse the knowledge a teacher needs to teach a specific mathematical topic, building on the notions of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) introduced by Shulman (1986, 1987).

This discussion leads me to analyse this framework through the lens of Chevallard's theory, in particular the theory of didactical transposition (Chevallard 1985, 1991) and the anthropological theory of didactics (Chevallard, 1999), and consequently to elaborate a new framework about knowledge for teaching mathematics. This framework will be applied to limits of functions in Chapter 4.

This chapter is therefore structured as follows:

3.1. What kind of knowledge does a mathematics teacher need?

3.2. SMK and PCK - a critical analysis

3.3. Mathematics for teaching and the didactical transposition

### **3.1 What kind of knowledge does a mathematics teacher need?**

There is a general consensus that a teacher needs to have solid mathematical knowledge in order to teach effectively. However, ideas differ about:

- What "solid mathematical knowledge" means for teachers;
- What other knowledge mathematics teachers need;
- How they acquire this knowledge.

In this chapter I address the first two issues, which I summarise in one question: what kind of knowledge does a mathematics teacher need? The question of how they acquire this knowledge will be discussed in Chapter 5.

Describing the kind of knowledge needed by mathematics teachers has given rise to many debates in the mathematicians and mathematics educators' community. I will summarise these debates through two papers: one by Boero, Dapueto &

Parenti (1996) entitled “Didactics of mathematics and the professional knowledge of teachers” and one by Cochran-Smith & Lytle (1998) entitled “Relationships of knowledge and practice: teacher learning in communities”.

### **3.1.1 Three main orientations (Boero, Dapueto & Parenti, 1996)**

Boero, Dapueto & Parenti consider three extreme orientations in mathematics teacher education, according to what knowledge is considered necessary for a mathematics teacher to be a “good” teacher.

The first one corresponds to the traditional idea that “He who knows mathematics knows how to teach it” (1996: 1099). They remark that this idea is not popular among mathematics educators but is still common amongst mathematicians and mathematics teachers. A more sophisticated (and less popular) version of this conception considers that, in addition to mathematical knowledge, teachers should have some meta-mathematical knowledge, such as knowledge of the history of mathematics, epistemology of mathematics and philosophy of mathematics. According to these authors, experiences of teacher education developed in line with this more elaborated conception lead to

a dramatic contradiction between ordinary, technical education in mathematics (ensured through traditional, non-interactive technical lectures and exercises concerning isolated and specialized fields of today’s mathematics) and some courses or seminars where prospective teachers (or in-service teachers) get to know different epistemological perspectives. In short, this contradiction results in a ‘cultural varnishing’, with no practical, deep influence on professional choices and classroom activities. (1996: 1100)

It seems that teachers are not able to link and apply theoretical knowledge acquired through the study of these different subjects.

The second orientation considers that the teacher must develop his/her professional competence as an artist. According to this conception, also very popular amongst mathematics teachers and mathematicians, a good teacher must be able to face professional problems in a flexible way, like an artisan, or to create innovations, like an artist. Teaching is considered as an art: “A good mathematics teacher must master mathematics and be acquainted with the art of teaching”

(Boero et al., 1996: 1101). This ‘art’ can be learnt through a suitable apprenticeship, an idea which reduces (or even excludes) the importance of scientific knowledge about mathematics education in pre-service and in-service teacher education. Furthermore, to become an efficient artisan, strong personality requirements and psychological resources seem to be necessary. As a consequence “the lack of these attributes may result in an increasing de-motivation towards the profession of teacher” (1996: 1102).

This orientation presents classroom practice as a talent that a teacher may or might not have.

As a consequence of these two first orientations, in many teacher training courses, mathematics is taught in the same way as in other university courses. According to the first conception, this should be enough for the teacher to apply this knowledge in teaching. According to the second orientation, besides this formal knowledge, teachers should have personal qualities. In both conceptions there is not much room for knowledge developed in mathematics education.

While the first conception emphasises the importance of mathematical knowledge and the second one the importance of practice, the third conception acknowledges the importance of a well-balanced mixture of different subjects and of the development of both knowledge and skills (1996: 1102-3). This conception, which has been more and more extensively represented and applied, states that the teacher's professional competence must be grounded in different scientific domains (mathematics, sciences of education, and didactics of mathematics).

According to Boero et al., although mathematics educators and many teachers share the third conception, their opinions differ concerning:

- the institutional environment where knowledge and skills are developed (pre-service and/or in-service education? “In series” or “in parallel” subjects?),
- the proportion of the different subjects in mathematics teacher education,
- the methodology to develop professional competence, and
- the content of education in didactics of mathematics (1996: 1103-04).



All these conceptions seem to be based on a dichotomy between content and practice. They obviously share the basic principle that teachers should have a solid mathematical knowledge, but they differ on how to develop teaching skills. The first orientation considers that teachers do not need them; knowing mathematics is enough for teaching it. The second orientation regards teaching as an art to be developed in practice. The third one recognises that knowledge developed in several areas such as psychology, pedagogy, and didactics of mathematics should be used to develop teachers' professional competence.

In line with this third orientation, I consider that mathematics teachers should develop both mathematical knowledge and knowledge about the practice of teaching mathematics. As I said before, in this chapter I will not address the issue of how to develop this knowledge, but rather focus on the content of this knowledge, particularly the relation between the scholarly mathematical knowledge and the knowledge needed by a teacher in his/her daily practice.

### **3.1.2 Relation between knowledge and practice (Cochran-Smith & Lytle, 1998)**

Cochran-Smith & Lytle (1998) have looked at this relationship between knowledge and practice through various initiatives intending to promote teacher learning, analysing the image of knowledge underlying these programmes. They distinguish three main conceptions of teacher learning: “*knowledge-for-practice*”, “*knowledge-in-practice*”, and “*knowledge-of-practice*”.

The two first conceptions assume that there are two distinct kinds of knowledge for teaching: one formal, produced following the conventions of social science research, and one practical, produced in the activity of teaching itself.

The first conception, “*knowledge-for-practice*”, is based on the idea that knowing more leads to a more effective practice. It considers that the formal knowledge, produced by experts, is the foundation for improving practice. Teaching is understood as a process of applying received knowledge, and teachers are considered as knowledge users, and not generators. As a consequence, in pre-service teacher training, the centrepiece of the curriculum is a codified knowledge divided into two separated areas: content knowledge and subject-specific

pedagogy. The emphasis is on what, not on how, teachers learn what is already “known” (Cochran-Smith & Lytle, 1998: 253-262).

Although recognising the importance of knowledge of practice, this conception can be seen as a more elaborated version of the first orientation pointed out by Boero et al., adding to formal knowledge on mathematics and meta-mathematics formal knowledge on other domains, such as psychology, general pedagogy and subject-specific pedagogy. By learning formal knowledge developed in all these domains, teachers are expected to somehow link them in order to improve their practice.

In the “knowledge-*in-practice*” conception, the emphasis is on teachers’ practical knowledge. A basic assumption is that teaching is “an uncertain and spontaneous craft situated and constructed in response to the particularities of everyday life of schools and classrooms” (1998: 262). It is assumed that teacher learning comes from reflection and inquiry in and on practice. No clear distinction is made between knowledge generation and knowledge use, thought and action are considered to be strongly linked. As a consequence, good teaching can be coached and learnt through reflective thinking guided by an insightful facilitator.

The point is for teachers to consider and reconsider what they know and believe, to consider what it means to know and believe something, and then to examine and reinvent ways of teaching that are consistent with their knowledge and beliefs. (1998: 272)

Although recognising that teachers’ teaching skills can be developed, this conception meets the second orientation by Boero et al., in considering teaching as an art. However it seems to be a more elaborated view of the teacher as an artist. Teachers’ practice is not seen as evolving spontaneously because of intrinsic qualities of the teacher, but as evolving through critical reflection and discussion with colleagues. As a consequence, “collaborative arrangements that support teachers working together to reflect in and on practice are the major contexts for teacher learning in this relationship” (1998: 263). Many action research programmes are and have been based on this conception, as I will show in Chapter 5.

The third conception, “knowledge-*of*-practice”, considers that teachers play a central and critical role in generating knowledge of practice.

The knowledge-*of*-practice conception stands in contrast to the idea that there are two distinct kinds of knowledge for teaching, one that is formal, in that it is produced following the conventions of social science research, and one that is practical, in that it is produced in the activity of teaching itself. (1998: 273)

Teachers are considered as co-constructors of knowledge. Therefore, professional development needs to create opportunities for them to explore and question, not only their own knowledge (as in the knowledge-*in*-practice conception) but also others’ interpretations, ideologies and practices,

In the knowledge-*of*-practice conception of teacher learning, the central image is of teachers and others working together to investigate their own assumptions, their own teaching, and curriculum development, and the policies and practices of their own schools and communities. (1998: 279)

Inquiry communities, where teachers and other participants invent new forms and frameworks of analysis and interpretation, are considered as the central context for teacher learning to occur. This conception is reflected in the increasing use, in pre-service and in-service programs, of critical reflections, ethnographies, teacher research, and some action research, where “student teachers are guided to connect their own experiences to critical, cultural, political, and economic theories and studies” (1998: 283). According to this conception, theory and practice should be integrated.

Graven (2005) for example describes the PLESME project, where mathematical knowledge and mathematics pedagogical knowledge were intertwined: “PLESME focused on the development of mathematical meaning and pedagogical forms simultaneously” (2005:219). Using this two-year INSET project as an empirical field for her research, she investigated the nature of mathematics teachers learning within a community of practice (2005:207).

She argues that most of the literature on teacher development indicates a focus on teacher change. In the South-African context, the curriculum support materials call for “radical teacher change where old practice is completely replaced by new

practice”. This view of teacher change is disempowering for teachers (2005: 223). On the contrary, the PLESME programme was based on a conception of learning as a life-long process, where teachers were expected to build their own knowledge. I will come back to this project in Chapter 5, when explaining how I set up the new institution.

### 3.1.3 My perspective

The “knowledge-of-practice” conception resonates with my own research. In Chapter 2, I argue that the Mozambican didactic institutions’ relation to the limit concept could be more elaborated. However, in order to teach limits in a different way, teachers need to challenge the didactical transposition made by these institutions. This will be only possible if they construct a new knowledge about this concept, based on a deep analysis of the mathematical concept, but also of the practice of teaching this concept in Mozambican secondary schools. This analysis could be illuminated by the results of research in mathematics education on the teaching of the limit concept, done in Mozambique or in other countries and adapted to the Mozambican context.

The importance of context has been stressed by Adler (2002). She observes: “Much of the teacher development literature is framed by countries whose historical trajectories in education and teacher education are very different from those of South Africa” (2002: 2). The same can be said of Mozambique, whose context is also very different from developed countries where most of the research in mathematics education is done.

For Adler the central issue is how subject knowledge, pedagogic subject knowledge and wider educational knowledge should be integrated in pre-service and in-service programmes (2002: 3). This integration faces two main tensions. The first one, the *subject-pedagogy tension*, is “how to integrate further learning of the subject with learning about how students in school acquire subject knowledge” (2002: 4). The second, the *theory-practice tension*, “revolves around how to combine learning about teaching through a distancing process (“theory”) with learning through immersion in experience (“practice”) (2002: 5).

In this chapter I address the first question: how to integrate mathematical and pedagogical knowledge on limits of functions: the different facets of the mathematical concept, the difficulties students face when learning this concept, the way it is taught in Mozambican secondary schools, and the different ways it could be taught. Obviously, other mathematics educators have already challenged this dichotomy between content and practice, and I will refer to them later in this chapter (see section 3.2.3). However, when I began this research, if general theoretical ideas had been developed on the necessity of integrating mathematical and pedagogical knowledge, little work had been done in practice on the components of this knowledge, and I had to make clear for myself what should be the components of the knowledge on limits that I wanted the teachers to develop in order to help them in their teaching practice. Although Even's framework is based on a distinction between Subject Matter Knowledge and Pedagogical Content Knowledge, it presents a rather integrated view of mathematical knowledge and practice of mathematics teaching. I used this framework to analyse the mathematical knowledge for teaching limits of functions in Mozambican secondary schools. My analysis of this framework evolved over time as I went into Chevallard's anthropological theory of didactics in more depth, and also through my work with the teachers involved in my research group.

One important question that emerged from this analysis is: What is teachers practice, or what are the teachers practices? Much of the research on teachers' practice in mathematics education address issues of classroom practice, such as learner-centred approaches, linguistic practices in the classroom, or gender issues in mathematics classroom. I personally consider two main parts of teachers' practice. The first one takes place when the teacher prepares his/her lessons. It is generally an individual activity, where the teacher can consult official documents such as the syllabus and the national exams, and other documents such as textbooks and worksheets. S/he also uses his/her knowledge of students' previous knowledge and difficulties, using his/her own experience as a student and as a teacher, or consults more experienced colleagues. The second part of a teacher's practice is actual classroom practice, where the teacher is in contact with students, trying to implement what s/he planned to do, but has to adapt it according to

students' reactions: their understanding, the questions they ask, how they solve the tasks, their mistakes. S/he can then decide to give more explanations, introduce a new task, and even go back to another topic.

In this study, I will mainly look at the first part of a teacher's activity: the preparation of the mathematical organisation. What knowledge does a teacher need when planning his/her lessons on a specific topic, taking into account the mathematical knowledge to be taught, the students' previous knowledge and difficulties, and the institutional constraints? In the next section I present an analysis of this knowledge through Even's framework and other researchers' work as it evolved during the research process, and as I linked it with Chevallards' anthropological theories of didactics.

### **3.2 SMK and PCK – a critical analysis**

Several authors have constructed frameworks to analyse teachers' mathematical knowledge. Most of these studies are based on the framework elaborated by Shulman (1986, 1987), who distinguishes three domains for teachers' knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge.

Even (1993) considers teachers' knowledge about a mathematical topic as having two main components: teachers' subject-matter knowledge (SMK) and pedagogical content knowledge (PCK). She states that a few years ago, teachers' subject-matter knowledge was defined in quantitative terms but that,

in recent years, teachers' subject-matter knowledge has been analysed and approached more qualitatively, emphasising knowledge and understanding of facts, concepts and principles and the ways in which they are organised, as well as knowledge about the discipline. (1993: 94)

#### **Pedagogical-content knowledge**

is described as knowing the ways of representing and formulating the subject matter that makes it comprehensible to others as well as understanding what makes the learning of specific topics easy or difficult. (1993: 94-95)

In this framework, Even (1993) distinguishes two kinds of knowledge, as in the knowledge-*for*-practice conception. She builds an analytic framework of SMK for teaching a specific topic in mathematics, which she applies to the study of the function concept. She notices that teachers' SMK and PCK are strongly interrelated, even though "there is little research evidence to support and illustrate the relationships" (1993: 95). In fact I will show that SMK, as defined in this framework, is not only formal knowledge, but has some practical teaching aspect that, in my point of view, could be considered as part of PCK. This reinforces my idea that SMK and PCK are intertwined and should be integrated in what Adler, Ball and Bass call *mathematics for teaching* (see section 3.2.3).

I will now present Even's framework, with some comments and comparisons with other mathematics educators' points of view, keeping in mind the topic "limits of functions".

### **3.2.1 Subject Matter Knowledge (SMK)**

In her analytic framework of subject matter knowledge (SMK) for teaching a specific topic in mathematics, Even (1990) distinguishes seven aspects that seem to form the main facets of this knowledge: essential features, different representations, alternative ways of approaching the concept, the strength of the concept, basic repertoire, knowledge and understanding of the concept, and knowledge about mathematics.

#### ***Essential features***

According to Even, this aspect of a concept "deals with the concept image, paying attention to the essence of the concept". (1990: 523)

In fact, in this statement, we should distinguish two aspects. The first one is epistemological: the "essence of the concept" is an intrinsic feature of this concept, or at least the features of this concept that are socially accepted. The second aspect is cognitive: the "concept image" is held by an individual. It can match or mismatch the essence of the concept.

The notion of concept image comes from Tall & Vinner, who introduced it as describing "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (1981: 152).

Another aspect of a concept is the concept definition, which is defined as a

form of words used to specify a concept. It may be learned by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. (1981: 152)

Resnick & Ford (1984), quoted by Even, present

“correspondence” - the match of one's subject mental picture of a specific concept with the correct mathematical concept - as an important criterion for evaluating well-structured knowledge about mathematics. (Even, 1990: 523)

According to Even, it is generally considered that “teachers should have a good match between their understanding of a specific mathematical concept they teach and the ‘correct’ mathematical concept” (1990: 523), but it seems rather difficult to define what exactly is meant by the ‘correct’ mathematical concept.

In Chevallard's words, the “correct mathematical concept” would be the concept as it can be found in the scholarly mathematical knowledge. Teachers learnt the mathematical concepts they have to teach through transpositions made by the institutions where they met these concepts, usually school in the first place, secondary school in the case of limits of functions, and then university. The way a concept is taught at university level is supposed to be closer to the scholarly mathematical concept than at secondary school. It should be the reference for the “correct mathematical concept”.

Even suggests that the good match between a teacher's conception and the ‘correct’ mathematical concept can be seen as being able “to judge whether an instance belongs to a concept family by using an analytical judgement as opposed to a mere use of a prototype judgement” (1990: 523). She argues: “it is not enough that teachers are able to distinguish between concept examples and non-examples when the instances match their concept image only” (1990: 524). In class, the teacher can find him/herself in a situation where s/he has to lead with unfamiliar examples. S/he must be able to distinguish between concept examples and non-



examples in these non-standard situations. Besides, many mathematical concepts evolve over time because of new mathematical knowledge, which leads to changes in the concept's definition. Teachers can be constrained by a limited and underdeveloped concept image.

In terms of didactical transposition, I would locate the essential features of a concept in scholarly mathematical knowledge. I will show in Chapter 4 that the limit concept evolved over time, with different essential features of this concept (static, dynamic, or operational) being stressed by the mathematicians' community in different historical moment of its development.

We have already seen that in Mozambican didactic institutions most of the tasks about limits were algebraic. What then are possible consequences of this focus on algebraic tasks for teachers' concept image of limits? What are possible consequences for their teaching of the limit concept? What essential features of the limit concept should teachers be aware of? These are fundamental questions that I will consider in the next chapter.

### ***Different representations***

Even states that

Teachers need to understand concepts in different representations, and be able to translate and form linkages among and between them. Different representations give different insights which allow better, deeper, more powerful and more complete understanding of a concept. (1990: 524)

Other mathematics educators have also pointed out the need for using several representations when teaching mathematics (Douady, 1986; Duval, 1996; Janvier, 1987). But what exactly are different representations?

The use of different representations can be seen from different points of view. I will focus on the epistemological point of view and on the cognitive point of view, referring to the distinction as illuminated by Duval.

The cognitive approach looks at how knowledge works from the angle of its mechanisms and processes as an individual's activity. The epistemological approach looks at the knowledge related to a specific field of objects, to their historical development and to their validation processes. (Duval, 1996 : 353)

[L'approche cognitive s'intéresse au fonctionnement de la connaissance sous l'angle des mécanismes et des processus qui la permettent en tant qu'activité d'un être individuel. L'approche épistémologique envisage les connaissances relativement à un domaine particulier d'objets, à leur développement historique et aux démarches de validation.]

### **Epistemological point of view**

Adopting an epistemological point of view, Douady defines “setting” as follows:

A setting is made up by the objects of a mathematical branch, by the relations between these objects, by their possibly different formulations, and by the mental images associated with these objects and relations.

[un *cadre* est constitué des objets d'une branche des mathématiques, des relations entre les objets, de leurs formulations éventuellement diverses et des images mentales associées à ces objets et ces relations ] (1986: 11)

For example, the geometrical setting includes geometrical objects (such as straight lines, segments, squares, angles, but also perimeters, areas), the relation between these objects (for example a square is a quadrilateral with four equal sides and right angles). To these objects and relations are associated mental images, and also representations. For example a square is usually represented by a figure similar to this one:



Douady also introduces the shift between settings as

a way to obtain different formulations for the same problem. These formulations are not necessarily exactly equivalent but allow a new access to the difficulties and the use of tools and techniques that were not obvious in the first formulation.

Translations from one setting to another often lead to unknown results, to new techniques, and to the creation of new mathematical objects. In short they enrich the original setting and the auxiliary working settings.

[Le changement de cadre est un moyen d'obtenir des formulations différentes d'un problème qui, sans être nécessairement tout à fait équivalentes, permettent un nouvel accès aux difficultés rencontrées et la mise en œuvre d'outils et de techniques qui ne

s'imposaient pas dans la première formulation. Les traductions d'un cadre dans un autre aboutissent souvent à des résultats non connus, à des techniques nouvelles, à la création d'objets mathématiques nouveaux, en somme à l'enrichissement du cadre origine et des cadres auxiliaires de travail.] (1986: 11)

This is what mathematicians do when they are solving a problem. The history of the limit concept gives us examples on how a concept can be studied in different settings. In the next chapter, I will refer to them as they historically appeared in mathematicians' works along time: the geometrical setting, the numerical setting, the formal setting, the algebraic setting, and the topological setting.

### **Cognitive point of view**

According to Duval, the use of semiotic representations is an intrinsic feature of cognitive functioning.

The semiotic representations are representations whose production depends on the calling up of a semiotic system. Thus the semiotic representations may be discursive productions (natural language, formal language) or non discursive productions (figures, graphs, diagrams, etc.). This production does not only meet a communication function: it can also come up to an objectivation function (for oneself) or a processing function.

[Les représentations sémiotiques sont des représentations dont la production ne peut pas se faire sans la mobilisation d'un système sémiotique : ainsi les représentations sémiotiques peuvent être des productions discursives (en langue naturelle, en langue formelle), ou non discursives (figures, graphiques, schémas, ...). Cette production ne répond pas uniquement ou nécessairement à une fonction de communication : elle peut aussi ne répondre qu'à une fonction d'objectivation (pour soi) ou à une fonction de traitement.] (1996: 356)

According to Duval, the use of semiotic registers is essential in mathematical activity, because of the “paradoxical character of mathematical knowledge”: “there is no other way of gaining access to the **mathematical objects** but to produce **some semiotic representation**” (1999: 1). Duval argues that to access a mathematical object, it is necessary to use at least two different semiotic registers. Otherwise the subject can mistake the mathematical object for its semiotic representation and “the understanding of mathematics requires not confusing the mathematical objects with the used representations” (1999: p.1).

Furthermore the representations in different semiotic registers need to be strongly coordinated.

The development of each register is not sufficient. It is also necessary that the different registers held by the subject, or that the teaching strives for this person to learn (for example the algebraic writing), be coordinated. This coordination is a condition for a full understanding insofar as it is the condition for a real differentiation between mathematical objects and their representation: it is seen in the capacity of recognizing that two different representations are representing the same object.

[Il ne suffit pas qu'il y ait un développement de chaque registre. Il faut également que les différents registres dont le sujet dispose, ou que l'enseignement s'efforce de lui faire acquérir (par exemple celui de l'écriture algébrique), se coordonnent. Cette coordination est la condition pour la maîtrise de la compréhension dans la mesure où elle est la condition pour une différenciation réelle entre les objets mathématiques et leur représentation : elle se manifeste par la capacité de reconnaître dans deux représentations différentes des représentations d'un même objet.] (Duval, 1996: 365)

### **Relation between settings and registers**

Duval highlights the differences and the links between settings and registers.

A register is established in relation to a semiotic system [...]. A setting is established in relation to theoretical objects, in this case mathematical objects. A change of settings can occur without any change of register, and a change of register without any change of setting, because a setting can require calling up several registers.

[Un registre se détermine par rapport à un système sémiotique [...]. Un cadre se détermine par rapport à des objets théoriques, en l'occurrence des objets mathématiques. Il peut y avoir changement de cadre sans changement de registre et changement de registre sans changement de cadre, car un cadre peut exiger la mobilisation de plusieurs registres] (1996: 357)

I will take as an example the following task:

Determine  $\lim_{x \rightarrow +\infty} (x - \ln x)$ .

This task belongs to the algebraic setting and can be solved without using any other register, for example using the L'Hôpital rule as follows.

$\lim_{x \rightarrow +\infty} (x - \ln x)$  leads to the indeterminate form  $[+\infty - \infty]$ . We can change the algebraic representation as follows:

$$\lim_{x \rightarrow +\infty} (x - \ln x) = \lim_{x \rightarrow +\infty} x \left( 1 - \frac{\ln x}{x} \right) = \left( \lim_{x \rightarrow +\infty} x \right) \cdot \lim_{x \rightarrow +\infty} \left( 1 - \frac{\ln x}{x} \right),$$

which leads to the indeterminate form  $\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = \left[ \frac{+\infty}{+\infty} \right]$ .

This indeterminate form can be solved using L'Hôpital's Rule:

$$\lim_{x \rightarrow +\infty} \frac{\ln x}{x} = \lim_{x \rightarrow +\infty} \frac{1/x}{1} = 0^+.$$

We then will have  $\lim_{x \rightarrow +\infty} x \left( 1 - \frac{\ln x}{x} \right) = +\infty$ .

The task has been solved by algebraic transformations, which obscures the meaning of this limit. What can we do to give some more meaning to this limit?

One of the possibilities would be to shift to the numerical register. We could give some values to the  $x$ -variable and calculate the corresponding  $y$ -values ( $y = x - \ln x$ ). We can present these calculations through a table.

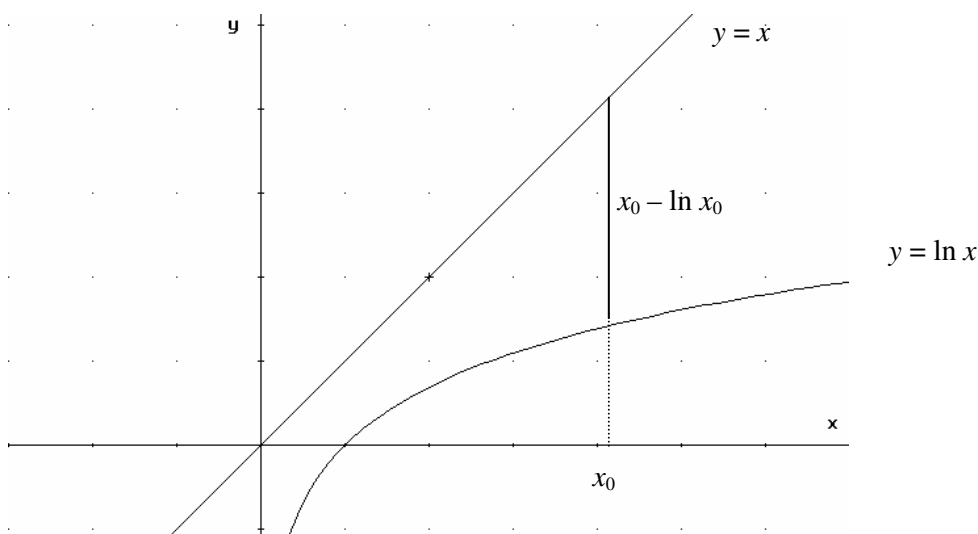
$x$	10	100	1000	10000
$y = x - \ln x$	7.697414907	95.39482981	993.0922447	9990.789666

The observation of this table allows us to make some conjectures:

- when  $x$  increases,  $y$  also increases;
- when  $x = 10^n$ , with  $n \in \mathbb{N}$ , it seems that  $y > 10^n - 10$ ;
- as a consequence the limit should be infinite.

The problem remains an algebraic problem, but the use of the numerical register helps us understand the meaning of the limit. We made a shift of registers and not a shift of settings.

Another way of giving sense to this limit is using a graphical register. Let's sketch the graphs of the functions  $f(x) = x$  and  $g(x) = \ln x$ .



**Figure 3.1 Limit of a function – graphical register**

For any value  $x_0$  of  $x$ , the difference  $x_0 - \ln x_0$  is the measure of a vertical segment linking the graphs of the two functions for this value, as shown on the graph above. On this graph, it is clearly apparent that when  $x$  increases, the graphs of the two functions are growing more and more distant. As a consequence, the limit should be infinite. The shift of semiotic register allows us to visualise the limit we want to evaluate.

As a conclusion, we can say that a mathematical object can be studied in different settings and using different semiotic registers. It is important that teachers know several representations<sup>5</sup> of a concept and are able to shift from one semiotic register to another, within the same setting, or from one setting to another. By doing so, they have access to new information about the concept and can construct a deeper understanding of it.

For example, the limit concept can be studied in a very formal way, using the  $\epsilon$ - $\delta$  definition, but can also be studied in a very intuitive way. Students can use numerical values in order to approach a limit and understand what it means that the limit of a function has a finite value or is infinite. They can also use a graphical register, in order to understand what it means for the graph that the limit of a function has a

<sup>5</sup> I will use 'representation' as a generic word to indicate either a setting or a register

finite value or is infinite. The limit concept can also be handled algebraically, calculating limits by using algebraic techniques. All these different aspects of limits of functions are complementary and changes of semiotic registers would help students to form a deeper understanding of this concept and more flexibility in their knowledge.

Therefore teachers should have a deep knowledge of the different representations in which the limit concept can be studied in order to organise the practical block with tasks in different settings or using different registers, and tasks designed to shift from one representation to another.

### ***Alternative ways of approaching the concept***

There are several ways of approaching a topic. Even argues that

there is a need to make good choices between different alternative approaches.  
Teachers should be familiar with the main alternative approaches and their uses.  
(1990: 525)

In fact, introducing the concept is the start of the second step of the didactical transposition, and a mathematics teacher has to choose a way to put his/her students in contact with the new concept. Chevallard (1999) also emphasizes the importance of what he calls the “first encounter” with a mathematical organisation. He distinguishes two main possible ways of organising this first encounter.

The first one is through a “cultural-mimetic problematic”, where the new object of knowledge is presented as already existing in some social practice (1999: 251). In this case, the student is required to work with this object by imitating the practitioner. Chevallard adds:

In its more demanding form, the cultural-mimetic encounter ought to lead [the student] to search and explain – in a discursive mode – the “*raisons d’être*” of this object, that is the reasons why this object has been created, or at least why it still remains in the culture.

[Dans sa version la plus exigeante, la rencontre culturelle-mimétique conduit en principe à rechercher et à expliciter – sur le mode discursif – les raisons d’être de

l'objet ainsi rencontré, c'est-à-dire les motifs pour lesquels cet objet a été construit, ou pour lesquels, du moins, il persiste dans la culture.] (1999: 251)

For example, introducing the limit concept through its formal definition is a cultural-mimetic encounter with this concept. It considers that the limit concept exists and is defined by mathematicians through a definition that the teacher communicates to students. They are required to work with this object. As it is difficult for secondary school students to work with the formal definition of limits, in this case the alternative is working with calculations, as shown in Chapter 2.

The second way of introducing a new mathematical organisation is the “in-situ encounter”, where the student, alone or with a group, is confronted with a task where the object at stake is expected to appear as necessary to answer one or more specific questions (1999: 251). Several mathematics educators have experimented with different ways of putting students in contact with the limit concept through activities where it comes into existence for students as a necessity to solve a task (Robinet, 1983; Cornu, 1984; Sierpinska, 1987; Schneider, 2001). I will describe them in Chapter 4.

Students' first encounter with a concept can have great implications for their concept image. Chevallard emphasizes that

if, quite obviously, the first encounter does not fully determine the relation to an object [...] it usually strongly orientates the future development of institutional and personal relations to this object.

[si, à l'évidence, la première rencontre ne détermine pas entièrement le rapport à l'objet [...] elle oriente en général fortement le développement ultérieur des rapports institutionnel et personnel à l'objet rencontré.] (1999: 252)

Mathematics teachers need to be aware of the importance of students' first encounter with a concept, and know several ways of introducing this concept, to be able to analyse the influence of these different approaches on students' conceptions. In that way they will be able to choose one or more of these approaches for their lessons.



Although Even considers the knowledge about alternative ways of approaching a concept as a part of SMK, I would rather consider that it belongs to PCK, as "knowing the ways of representing and formulating the subject matter that make it comprehensible to others" (Even, 1993: 94-95). As part of teachers' activities when setting up the knowledge actually taught in the classroom, organising students' first encounter with a concept does not require only scholarly mathematical knowledge (SMK), but also mathematical knowledge oriented to teaching (PCK). The teacher not only needs to know alternative ways of introducing the concept, but also be able to analyse students' difficulties and possible consequences of each alternative for students' concept image. SMK and PCK are indeed interrelated.

As a conclusion, knowing several ways of introducing a concept is, in fact, a very important component of teachers' knowledge of a concept. In the next chapter, I will present different ways of introducing the limit concept found in textbooks, or in research papers where they are part of didactical engineering experiments.

### ***The strength of the concept***

Even argues that

concepts become important and powerful because there is something special about them which is very unique and opens new possibilities. Teachers should, therefore, have a good understanding of the unique powerful characteristic of the concept. (1990: 525)

According to Arsac, the problem of justifying the contents of teaching is a fundamental point highlighted by didactical transposition theory (1992: 108).

It would appear that different justifications are possible: teaching content may refer to social practices, professional or domestic ones in particular, or to scientific knowledge. (1992a: 109)

Teachers should be aware of the reasons why a specific content is part of the syllabus of a specific class in a specific institution. It is what Chevallard calls the “social justification”<sup>6</sup> for teaching specific content

For example, some years ago “historical and dialectical materialism” was taught as a subject in all university courses in Mozambique. The justification for that was the political orientation of the Mozambican government. This subject now has no place in Mozambique because the political orientation of the government that legitimated this teaching has changed.

In the first presentation of didactical transposition theory, Chevallard “insists on the priority of scholarly mathematical knowledge as a reference for the teaching of mathematics” (Arsac, 1992a: 109). The gap between the scholarly mathematical knowledge and the knowledge to be taught can be explained by the existence of factors weighing on the institution, in particular the age of the learners and time management.

Teachers should know the social justification for teaching a specific concept, which includes the strength of this concept, particularly its relationship with other mathematical concepts, even if these concepts are not taught at secondary school level and students will only access them later, for instance at university. I will show in the next chapter that limit concept is a very powerful concept, a basic concept for the study of calculus, and has many applications in different areas.

### ***Basic repertoire***

For Even, the basic repertoire of a mathematical topic or concept

includes powerful examples that illustrate important principles, properties, theorems, etc. Acquiring the basic repertoire gives insights into and a deeper understanding of general and more complicated knowledge. (1990: 525)

She argues that “only if the basic repertoire is acquired meaningfully and with understanding can it be used appropriately and wisely” (1990: 525). In

Chevallard’s terms, the basic repertoire relates to the practical block. In fact, one of the activities of a teacher when teaching a concept is to organise students’

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<sup>6</sup> In that case *social* means justification *for the society* and does not indicate the nature of the justification, which can be for example epistemological, political, professional, or social.

tasks. S/he needs to have a basic repertoire, from where s/he will select tasks, in accordance with the syllabus, the age and previous knowledge of the learners, and institutional constraints such as the time and the means that are available (for example calculators, graphic calculators, computers).

According to Even, teachers often use the same repertoire of examples and tasks, without reflecting on the possibility of using different examples and creating new tasks, which could give access to a deeper understanding of the concept. Looking at two Mozambican didactic institutions, we already saw that the practical block for the study of limits of functions was almost limited to algebraic tasks. Other kinds of tasks appear in the syllabus (graphical tasks) or in university level textbooks (word problems to apply the limit concept), but it seems that they were not selected by teachers when planning their lessons. This practical block needs to be broadened. Does a teacher's basic repertoire on limits of functions enable him/her to do that?

As for the ways of approaching a new concept, this category can be seen as belonging at the same time to SMK and to PCK. It is grounded in a deep mathematical knowledge, but relates to teaching and has strong links with different representations. If a deep understanding of the concept is reached through shifts between semiotic registers, a teacher's basic repertoire should include tasks that enable students to shift from one register to another, and to choose a "good" register to solve a task, depending on the task they have to perform.

### ***Knowledge and understanding of the concept***

Even describes conceptual knowledge as "knowledge which is rich in relationships. It is a network of concepts and relationships" (1990: 526). She points out that "school mathematics tends to over-emphasise procedural knowledge without close relation to conceptual knowledge and meaning" (1990: 526). Teachers who do not understand the connections between concepts and procedures "are not able to solve problems, or they may generate answers but not understand what they are doing" (1990: 527).

According to Janvier, understanding

- “implies a series of complex activities”;
- “presupposes automatic (or automatized) actions monitored by reflection and planning mental processes”;
- “is an ongoing process” and
- “is a cumulative process mainly based upon the capacity of dealing with an “ever-enriching” set of representations” (1987: 67).

As suggested by Janvier, dealing with different representations gives meaning to a concept. On the other hand, the knowledge of a concept’s different features helps to understand it better. Furthermore, the relationships between one concept and other concepts, which also play an important role in the understanding of this concept, are part of the strength of the concept.

This category of teachers’ knowledge seems different from the other categories defined in Even’s framework. While the previous categories relate to some activity of the mathematics teacher when performing the second step of the didactical transposition (organising the first encounter of students with some mathematical organisation, justifying the teaching of this organisation, organising a practical block by providing tasks in different settings and for shifting from one register to another), this category is related to the way the knowledge is held by mathematics teachers. This quality of teachers’ knowledge relates to what Ball, Bass & Hill call “connectedness” (2004: 59-60). They state:

Another important aspect of knowledge for teaching is its **connectedness**, both across mathematical domains at a given level, and across time as mathematical ideas develop and extend. Teaching requires teachers to help students connect ideas they are learning. [...] Teaching involves making connections across mathematical domains, helping students build links and coherence in their knowledge. (2004: 59-60)

It is indeed important that teachers be able to connect, and help their students connect, mathematical ideas. However, I do not consider connectedness as a category of teachers’ knowledge, but rather as a quality of this knowledge grounded on the knowledge defined in Even’s previous categories. For example, knowing alternative ways of introducing the limit concept, its various

representations, and the strength of this concept, should lead teachers to see it not as an isolated concept but as part of a network of concepts.

### ***Knowledge about mathematics***

According to Even, "Knowledge of a specific piece of mathematics includes more than conceptual and procedural knowledge. It also includes knowledge about the nature of mathematics" (1990: 527).

This category of knowledge relates to mathematics' disciplinary features, to what makes mathematics different from other subjects, such as physics or biology. It relates to scholarly mathematical knowledge in terms of the scientific methods used by mathematicians, such as defining and proving. These methods are also the object of didactical transposition, depending on the level of teaching. For example at primary school, learners are not usually required to define a square or to prove any of its properties, but only to recognise its shape and use its properties, for example calculate its area. At secondary school, students will be required to define a square, and to be able to prove some its properties. A mathematics university student should be able to give formal definitions and rigorous proofs. In order to make this transposition, teachers need a good knowledge of mathematics, but also a good knowledge of students' previous knowledge and difficulties.

Secondary school mathematics teachers need to introduce definitions to students. They should be aware of the role of definitions in mathematics. Ball et al. state that teachers need to appreciate "what a mathematical definition needs to do" (2004: 57). And they explain:

Mathematical definitions are precise statements of the nature of objects, procedures, and properties. They make it possible to be clear, and to communicate effectively. They also play a crucial role in supporting mathematical reasoning." (2004: 57)

Ball et al. also contend that

Knowing how definitions function, and what they are supposed to do, together with also knowing a well-accepted definition in the discipline, would equip a teacher for the task of developing a definition that has mathematical integrity and is also comprehensible to students. A definition of a mathematical object is

useless, no matter how mathematically refined and elegant, if it includes terms that are beyond the prospective user's knowledge (2004: 58).

I have already shown that, in the teaching of limits in Mozambican didactic institutions, even at the Pedagogical University, the formal definition was never used in practice. The  $\varepsilon$ - $\delta$  definition is often taught at secondary school, despite the teacher knowing that students will not understand it. Knowing several definitions of the limit concept, and being able to choose a definition adapted to students is an important part of the teacher's role.

Another important aspect of knowledge about mathematics is the role of proofs. Teachers need to introduce some theorems or properties and their proof, and to help students develop their ability of constructing strong arguments. On the other hand, a teacher needs to know why and in which domain a rule works, and be able to explain it to his/her students. This would also help him/her to analyse students' mistakes resulting from using a rule out of its domain of validity.

We have already seen that most of the rules to calculate limits were not proved in Mozambican secondary schools, and even at the Pedagogical University, and asked the question: As a result, what conception of the role of proofs in mathematics will be developed by teachers?

On the other hand, a teacher must know that to solve a task we often can use several methods. Ball et al. contend that

when teachers see methods they have not seen before, they must be able to ask and answer – for themselves – a crucial **mathematical** question: What, if any one exists, is the method, and will it work for all cases? (2004: 56)

Another aspect of the knowledge a teacher needs to have about mathematics is the use of notation. Symbols are often introduced in schools without explaining their meaning, how to use them, and how to read them. This is the case for example in Mozambican schools with the use of quantifiers. We will also see that students often have difficulties in using the limit symbol.

Obviously, a mathematics teacher should have a good knowledge about mathematics in general. From my point of view, there is a dialectic relationship between the knowledge about mathematics and the knowledge of a specific

mathematical topic. On the one hand, the knowledge about mathematics helps the understanding of a specific topic. On the other hand, the knowledge about mathematics is built on the deep knowledge of specific topics and the reflection on this knowledge. In this way, the knowledge teachers have about the nature of mathematics will influence their teaching of limits of functions, but they will also deepen their knowledge about mathematics by reflecting on the concept of limits of functions and its teaching.

### ***Overview***

In this section I presented the seven categories which, according to Even, make up the SMK needed by a teacher: essential features, different representations, alternative ways of approaching the concept, the strength of the concept, basic repertoire, knowledge and understanding of the concept, and knowledge about mathematics. Some of these categories have been further elaborated by other researchers, and I referred to some of these studies. Relating them to the limit concept, I showed that most of them were pertinent, and should be part of the abilities that a teacher needs to teach a specific concept in schools. Nevertheless, this classification is not well theorised, as these categories mix epistemological and cognitive aspects, the mathematical and the pedagogical:

- The category “essential features” refers both to the scholarly mathematical concept (epistemological) and to the concept image (cognitive).
- Some of the categories considered by Even as belonging to SMK seem to be strongly related to teaching, and could be considered as part of PCK. For example, the alternative ways of approaching the concept refers to teaching in the classroom, as well as the basic repertoire.

This framework failed in separating the mathematical (SMK) and the pedagogical (PCK). In fact, if we consider that all these seven categories belong to SMK, we have to ask the question: What then is Pedagogical Content Knowledge?

### 3.2.2 Pedagogical Content Knowledge (PCK)

Even describes PCK as

knowing the ways of representing and formulating the subject matter that make it comprehensible to others as well as understanding what makes the learning of specific topics easy or difficult. (1993: 94-95)

Looking at teachers' pedagogical content of geometry and referring to Shulman, Rossouw & Smith describe PCK as “a means to identify teaching expertise which is local, part of the teachers' personal knowledge and experience” including

(a) the different ways of representing and formulating the subject matter to make it comprehensible to others, (b) understanding what makes the teaching of specific topics easy or difficult and (c) knowing the conceptions and pre-conceptions that learners bring to the learning situation (1998: 57-58).

They also referred to Marks (1990), who

has painted a portrait of PCK as composed of four major areas: (a) knowledge of subject matter, (b) knowledge of student understanding, (c) knowledge of the instructional process and (d) knowledge of the media for instruction (Rossouw & Smith, 1998:58).

Some of the categories identified as belonging to the PCK, and related to a specific topic, have already been considered in the description of SMK (according to Even's framework), for example different ways of representing the concept and knowledge of the subject matter. The new and important point, introduced by these authors, deals directly with students' conceptions and difficulties: “knowledge of students understanding”, “understanding what makes the teaching of a specific topic easy or difficult” and “knowing the conceptions and pre-conceptions of the learners”. This is a critical component of PCK, and it emerges several times in the previous analysis of SMK components, showing that this facet of teachers' knowledge about the learning of a topic, and specifically of the limit concept, should be considered as a category of teachers' knowledge to be developed.



### 3.2.3 SMK versus PCK

This study of teachers' knowledge of limits of functions through Even's framework shows that most aspects considered in this framework are relevant, being part of the knowledge that a teacher needs to teach a specific topic. However this classification does not appear systematic. Two categories strongly refer to SMK or, in Chevallard's words, to *scholarly mathematical knowledge*. They are the strength of the concept, and knowledge about mathematics. Four other categories can be seen as belonging both to SMK and PCK, as they both refer to mathematical and pedagogical knowledge. They are essential features, linked to students' concept image; different representations, alternative ways of approaching the concept, and basic repertoire, all of which refer to teaching practice. The seventh category is of a different nature, as it refers to the quality of teachers' knowledge and not exactly to the content of this knowledge.

This leads me to ask the question: Why separate these two aspects, SMK and PCK?

Several researchers have challenged the distinction between SMK and PCK.

Cochran, DeRuiter & King, from a constructivist point of view, expanded the notion of PCK, by placing "emphasis on knowing and understanding as active processes and on simultaneous development of all aspects of knowing how to teach" (1993: 263). They introduce the notion of *pedagogical content knowing* (PCKg) as teachers' understanding of four components: pedagogy, subject matter content, student characteristics, and the environmental context of learning (1993: 266).

Steinbring notices that the description of content knowledge made by Shulman

relates to the scientific discipline and to academic knowledge but gives no specific attention to the needs regarding the teaching and learning of this subject matter knowledge (1998: 157)

And he asks the question: How do SMK and PCK relate to one another?

Steinbring observes that, in a linear model of the teaching-learning process, this process is seen as divided into two steps: first the teacher "prepares knowledge for

mathematics teaching”, and then s/he “conveys mathematical knowledge to the students” (1998: 158). He states:

According to this model, mathematical content knowledge is primarily needed during the first step in this process, whereas pedagogical content knowledge is necessary for the conditions and forms of the transmission of school mathematical knowledge to students during the second step (1998: 158).

Steinbring argues that, in practice, these two steps are not separated, and the teaching-learning process can be seen as two autonomous systems that influence one another: the student’s learning process and the interactive teaching process between teacher and students (1998: 158).

In this second model, pedagogical content knowledge does not primarily serve to organise the transmission of mathematical content knowledge: here, a new type of professional knowledge for mathematics teachers is needed – a kind of a mixture between mathematical content knowledge and pedagogical knowledge (1998: 159).

In fact, if we can distinguish two moments in the teacher’s work, the moment where s/he prepares his/her lesson and the moment where s/he actually teaches in the classroom, these two moments influence each other. When preparing his/her lesson, a teacher must take into account his/her learners, their age, their previous knowledge, their difficulties, his/her relation with the class, as well as the teaching conditions and constraints inside the institution. On the other hand, when teaching s/he must take several decisions which depend on the learners’ performance and behaviour. S/he can add or eliminate some task, give more explanation, or even come back to another topic that emerged from the students’ questions. This teaching experience will also inform a new preparation on the same topic to teach it in another class or another year. Therefore, and in line with Steinbring, and as already mentioned before, I would rather consider the professional knowledge needed by mathematics teachers as a mixture between SMK and PCK. From my point of view, SMK and PCK are not only interrelated, as stated by Even (1993), but strongly intertwined and articulated, in a way that in practice it is quite difficult to distinguish one from the other. Teachers’ learning of mathematics must be oriented to teaching.

This point of view is also supported by Ball et al., who argue that knowing mathematics for teaching “requires a unique understanding that intertwines aspects of teaching and learning with content” (2004: 54). They assert that, instead of investigating what teachers need to know by looking at what they need to teach, we should focus on the work that they do. And they ask the question: “What do teachers do, and how does what they do demand mathematical reasoning, insight, understanding, and skill?” (2004: 54).

They suggest that

Teachers’ opportunities to learn mathematics should include experiences in unpacking familiar mathematical ideas, procedures and principles. But [...] learning mathematics for teaching must also afford opportunities to consider other aspects of proficiency with mathematics – such as understanding the role of definitions and choosing and using them skilfully, knowing what constitutes an adequate explanation or justification, and using representations with care. Knowing mathematics for teaching often entails making sense of methods and solutions different from one’s own, and so learning to size up other methods, determine their adequacy and compare them, is an essential mathematical skill for teaching [...] (2004: p. 64)

Many of the skills indicated by these authors were also present in Even’s categories, in particular in knowledge about mathematics and different representations. The new important idea from this quote is the concept of ‘unpacking’ familiar mathematical ideas. Without unpacking their mathematical knowledge of a concept, teachers will only be limited to reproduce the ways they learnt this concept in schools or at university.

I already referred to Adler arguing that subject knowledge, pedagogic subject knowledge, and wider education knowledge should be integrated in pre-service and in-service programmes (2002:3). In line with this position, the Quantum project in South-Africa (Adler, 2004; Adler, Davis, Kazima, Parker & Webb 2005; Adler & Davis, 2006) aims to elaborate *mathematics for teaching* (MfT), regarding

the mathematical work of teaching as a particular kind of mathematical problem-solving - a situated knowledge, shaping and being shaped by the practice of teaching (Adler et al., 2005: 2).

They also consider unpacking and decompression as a key element of knowing and doing mathematics in and for teaching (Adler & Davis, 2006: 2). Adler & Davis notice that in South Africa's teacher training,

compression or abbreviation of mathematical ideas dominates formal evaluation. There is a limited presence of interesting instances of unpacking or decompression of mathematical ideas as valued mathematical practice (2006: 271).

The study of the Pedagogical University's institutional relation to the limit concept highlights the same limitation in Mozambique. Student-teachers are only asked to solve routine tasks, without any further reflection on this concept. There is no "unpacking of mathematical ideas, procedures or principles", but rote reproduction of procedures.

For Long (2003), also from the Quantum project,

Subject matter knowledge and pedagogic content knowledge have been elaborated separately for theoretical purposes though we can see in practice that they are inextricable (2003: 8).

I already mentioned the PLESME project, where mathematical knowledge and mathematical pedagogical knowledge were intertwined (see pages 61-62)

In line with these mathematics educators, I want to develop the idea of a mixed content and pedagogical knowledge, looking at teachers' knowledge through the lens of the didactical transposition. In line with Adler, I will call this knowledge *Mathematics for Teaching* (MfT).

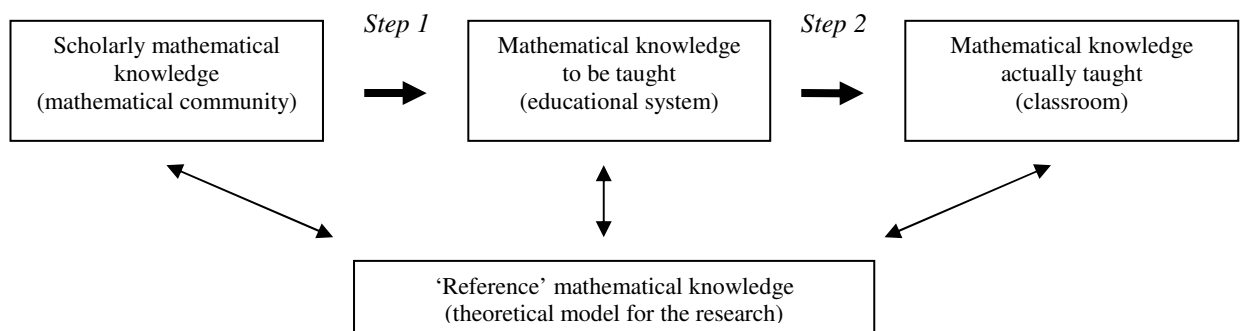
Most of the Mozambican teachers' personal relation to limits of functions has been shaped by Mozambican institutions, in particular Mozambican secondary schools and the Pedagogical University. We already saw that in these institutions most of the tasks about limits were algebraic tasks, which do not allow students to develop a deep concept image of limits. As a consequence they will probably consider the didactical transposition usually made in secondary schools as

transparent. What would these teachers need in order to break the rules, to question the way limits are taught in Mozambican secondary schools? I argue that in order to do that, teachers should be aware of the whole process of the didactical transposition. This means in the first place that they should be able to look critically at the first step of this transposition (scholarly mathematical knowledge → knowledge to be taught) already taken by the institution. They also should be able to analyse the possibilities of providing and enacting a second step (knowledge to be taught → knowledge actually taught) different from the one usually carried out in Mozambican secondary schools, despite the institutional constraints.

I will elaborate this idea in the next section, analysing the knowledge that a teacher would need to perform this task. As I already mentioned, I will not look at teachers' practice in the classroom but rather focus on the work done by the teachers when planning their lessons.

### 3.3 Mathematics for teaching and the didactical transposition

In the previous chapter, I presented the didactical transposition using the following diagram.



**Figure 3.2 The didactical transposition**

This diagram highlights the two main steps of the didactical transposition. The first step aims to select some contents in the *scholarly mathematical knowledge* and to convert them into *knowledge to be taught* in an institution, according to the age of the learners, but also to institutional constraints. This first step is carried

out by the institution, which must have a socially accepted justification to explain its choices.

The second step consists in converting these contents into *knowledge actually taught* in the classroom. This is the teacher's work. Using Chevallard's terms, this work can be seen as the *teacher's didactical praxeology* that, according to Bosch & Gascón, is

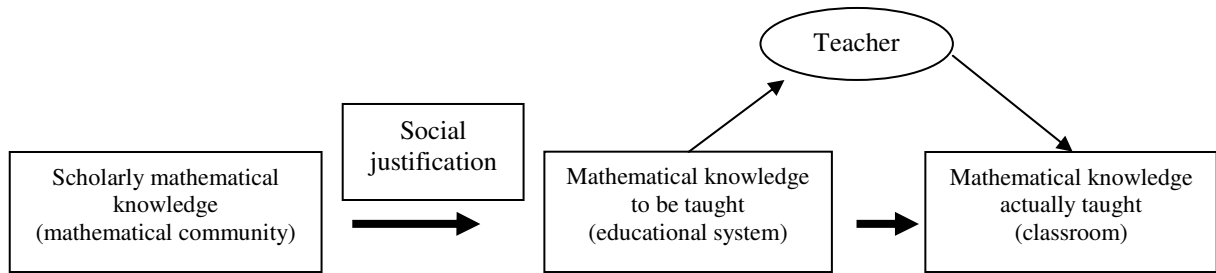
an institutionalised practice that, as any other practice, can be divided into a "practical" block and a discourse ("logos") which justifies, interprets, guides and modifies the practice.

[une pratique institutionnalisée qui, comme toutes les autres, peut se diviser en un bloc "pratique" et en un discours ("logos") qui justifie, interprète, guide et modifie la pratique.] (2002: 2)

These authors consider that the teacher's didactical praxeology has three main characteristics:

- It is empirical, because it exists in a specific institution at a specific moment, with specific characteristics and limitations.
- It is spontaneous, because the techniques used to solve the tasks are not necessarily organized by a technological-theoretical discourse; many of them are seen as natural inside the institution.
- It is the praxeology of that specific teacher, a result of his/her subjections to the several institutions where s/he has met the content at stake.

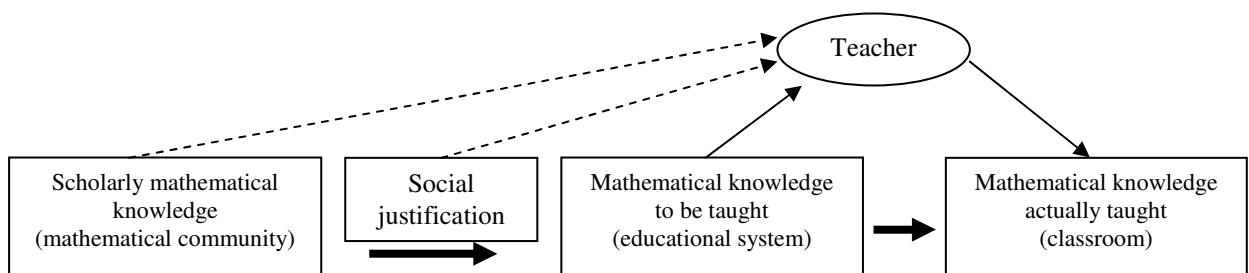
In fact, to prepare his/her classes about a specific topic, a teacher's references are usually the *knowledge to be taught*, which can usually be found in the syllabus, but also in the national examinations or in textbooks (when available) or worksheets (as happens in Mozambican secondary schools), as well as the experience of his/her own contacts with this topic through several institutions. I would then represent the traditional position of the teacher within the didactical transposition as follows (Figure 3.3). In this new diagram, I also indicate the social justification needed to perform the first step of the didactical transposition.



**Figure 3.3 Traditional location of the teacher in the didactical transposition**

In the process as described in Figure 3.3, the teacher would probably reproduce the didactical transposition usually carried out within the institution where s/he is teaching. Teachers whose personal relation to a concept has been shaped by a similar institutional relation are expected to be “good subjects” of the institution, and not to challenge the institutional relation.

How could a teacher break the institutional routine? I argue that, to be able to do that, a teacher needs to be aware of the whole process of the didactical transposition. This means on the one hand to be aware of the first step of this didactical transposition, which is usually considered as natural and, on the other hand, to be able to consciously take the second step of this transposition, not only looking forward to the classroom situation, but also looking backward to the scholarly mathematical knowledge and to the social justification in order to teach this specific knowledge in that specific institution and at this specific level. I would then represent the new location of teacher in the didactical transposition as follows (Figure 3.4.).



**Figure 3.4 New location of the teacher in the didactical transposition**

What kind of knowledge would enable a teacher to perform his/her tasks within this new location?

In the first place, s/he obviously ought to have sound knowledge of the *scholarly mathematical knowledge*. This firstly includes specific knowledge of the concept to be taught, for example the limit concept, in particular its essential features, its definition, the theorems and proofs which underline the work with this concept. In other words, s/he should have a good knowledge of the reference MO's knowledge block. But it also means that s/he must have more general mathematical knowledge, such as knowledge about the role of definitions, proofs, and symbols. Therefore, teachers' knowledge of the *scholarly mathematical knowledge* includes the following categories of Even's framework: essential features, knowledge and understanding of the concept, and knowledge about mathematics.

In the second place, the teacher should understand the social justification to teach this concept. S/he should understand why this concept has been chosen to be taught in that specific institution and at that specific level. This means that s/he must have a broad view of both the mathematical organisations taught in schools and at university level, and of the link between them.

Chevallard considers a hierarchy of levels for mathematical organisations: specific, local, regional, and global organisations (Chevallard, 2002b: 2). For example, a specific kind of limit and the algebraic technique used to evaluate this limit constitute a *specific organisation* [organisation ponctuelle]. It belongs to a *local organisation*, which could be, for example, the algebraic determination of limits, including several kinds of algebraic tasks and different techniques to solve them. In the same way, this local organisation is part of a *regional organisation*, which includes several local organisations sharing the same theory. For example, in this case, the regional organisation should be the study of limits. Finally, this regional organisation is included in a more *global organisation*, which can be identified with a domain of study, in that case mathematics. Chevallard contends that, when determining the mathematical organisations to be set up in their classes, teachers tend to rely only on the more specific levels: specific and local organisations (Chevallard, 2002b: 3). This can lead on the one hand to a lack of motivation of the tasks to be solved and, on the other hand to incomplete mathematical organisations. We already saw that this is what happens with limits



of functions in Mozambican institutions: students do not know why they have to learn this topic, why they have to calculate limits, and the theoretical part does not correspond to any practical block.

Chevallard argues that an essential principle of the ecology of didactical organisations should be the following:

In order to acknowledge what could be – and what cannot be – the organisation of the study of a topic, it is necessary to take into account the superior stages of the hierarchy of the mathematical determination levels.

[pour reconnaître ce que pourrait être – et ce que ne peut pas être – l’organisation de l’étude d’un sujet ou d’un thème donné, il convient de prendre en compte les échelons supérieurs de la hiérarchie des niveaux de détermination mathématique.]  
(2002b: 6)

This means that a teacher should understand the social justification for teaching a specific concept inside a specific institution. Why should students learn this concept at that level? How will they use it in their further studies? How will this concept be applied? How does it relate to other concepts? In Even’s words, what is the strength of this concept?

Then the teacher should be able to select contents for his/her classes. This means that s/he must build a *didactical organisation* (Chevallard, 2002a). To analyse how a didactical organisation allows the set up of a mathematical organisation, we can first look at the way the different moments of the study of this MO are settled in the classroom. Chevallard (2002a) presents a model of six moments of study, divided into four groups. The order of these moments is not a fixed one. Depending on the kind of didactical organisation, some of these moments can appear in a different order, but all will probably occur. They are the following:

First group (Study and research activities [SRA])

1. Moment of the *(first) encounter with* [the task]  $T$  ;
2. Moment of *exploration of*  $T$  and *emergence of the technique*  $\tau$  ;
3. Moment of construction of the *technological-theoretical block*  $[\theta/\Theta]$ .

Second group (Syntheses)

4. Moment of *institutionalisation*.

Third Group (Tasks and problems)

5. Moment *of working* the mathematical organisation (particularly *the technique*).

Fourth Group (Control)

6. Moment *of evaluation*.

[Groupe I (Activités d'étude et de recherche [AER])

1. Moment de la (*première*) *rencontre* avec  $T$  ;
2. Moment de *l'exploration de  $T$  et de l'émergence de la technique  $\tau$*  ;
3. Moment de la construction du *bloc technologico-théorique* [ $\theta/\Theta$ ].

Groupe II (Synthèses)

4. Moment *de l'institutionnalisation*.

Groupe III (Exercices et problèmes)

5. Moment *du travail* de l'organisation mathématique (et en particulier *de la technique*).

Groupe IV (Contrôles)]

6. Moment *de l'évaluation*.] (2002a: 12)

These moments of study can be organised in different ways and in a different order.

Using Chevallard's classification of the moments of study, I separated the teacher's main task (to teach a mathematical concept) into several smaller tasks, corresponding to these different moments. The teacher has to:

1. Introduce the concept to his/her students (first encounter);
2. Introduce some tasks and some techniques to solve these tasks (practical block);
3. Justify and explain these tasks and techniques through a technological discourse (knowledge block);
4. Make clear what students need to know (institutionalisation);
5. Organise students' work of the techniques;
6. Evaluate the students.

As I am focussing on the work done by the teacher prior to teaching, when planning his/her lessons, I will not consider the last task (evaluating the students), which takes place after and depends on the work with students within the classroom.

What knowledge does a teacher need to perform the other tasks, challenging the way they are usually performed inside his/her institution?

First of all, the teacher must organise his/her students' first encounter with the concept. In order to choose a suitable way to introduce the concept, s/he needs to know several different ways of doing that, but s/he also needs to know his/her students' conceptions about this concept and related concepts, as well as the difficulties students usually face when studying this concept. This is what Even calls "alternatives ways of approaching the concept".

Then, to help his/her students explore the concept, in order to develop a good concept image, the teacher must also be able to lead them to work with different semiotic representations. S/he must give them different kinds of tasks and lead them to use different techniques to solve these tasks, choosing a suitable technique for a specific task. This means that s/he needs to have a good knowledge of the different semiotic representations in which this concept can be studied, and an extended basic repertoire of tasks within these representations and to be able to shift from one representation to another. This relates to "different representations" and "basic repertoire" from Even's categories.

Then the teacher has to choose what technological elements s/he will give to his/her students, in order to justify and explain the techniques introduced to solve the tasks. Which definition of the concept should be given to students, according to their age and to their previous knowledge? Which theorems, which proofs can justify these rules? Are students able to understand these proofs? If not, how can these rules be explained? Can a shift of semiotic representation help explain these rules? Here again the teacher needs a good knowledge of the scholarly mathematical knowledge, but also of different representations and students' previous knowledge.

Therefore, I would classify the professional knowledge that a teacher needs to consciously perform the second step of the didactical transposition of a specific mathematical concept according to the following categories:

- (a) Scholarly mathematical knowledge of the concept; this includes definitions of the concept, properties of the concept and their proofs,

essential features, correct use of notations and symbols, as well as general knowledge about mathematics.

- (b) Knowledge about the social justification to teach this concept: this mainly relates to the strength of the concept.
- (c) Knowledge about how to organise the students' first encounter with the concept; this relates to alternative ways of introducing the concept.
- (d) Knowledge about the practical block of the MO (tasks and techniques); this includes different representations, and basic repertoire.
- (e) Knowledge on how to construct the knowledge block (technological elements to justify the techniques) according to learners' age and previous knowledge.
- (f) Knowledge about students' conceptions and difficulties when studying this concept.

These categories include most of Even's categories of the SMK, as well as the category "students' conceptions and difficulties" taken from the PCK. In fact, the teacher's knowledge about students' conceptions and difficulties must inform all the choices made by this teacher when selecting the knowledge to be taught. However these categories are defined in a more systematic way, looking at teachers' tasks when building a didactical organisation. The relation of these categories of *mathematics for teaching* to the didactical transposition and to Even's categories is summarised in Figure 3.5 (page 95).

*Mathematics for teaching* (Adler & Davis, 2006) as defined through these categories relates to the third orientation described by Boero et al. (1996), as it takes into account the development of both mathematical knowledge and knowledge of the practice of teaching. It can be seen as a kind of *knowledge-of-practice* (Cochran-Smith & Lytle, 1998), where mathematical and pedagogical knowledge are intertwined, and which aims to enable teachers to unpack mathematics (Ball et al., 2004).

In the next chapter, I use these categories to elaborate in more detail *mathematics for teaching* the specific topic: limits of functions.

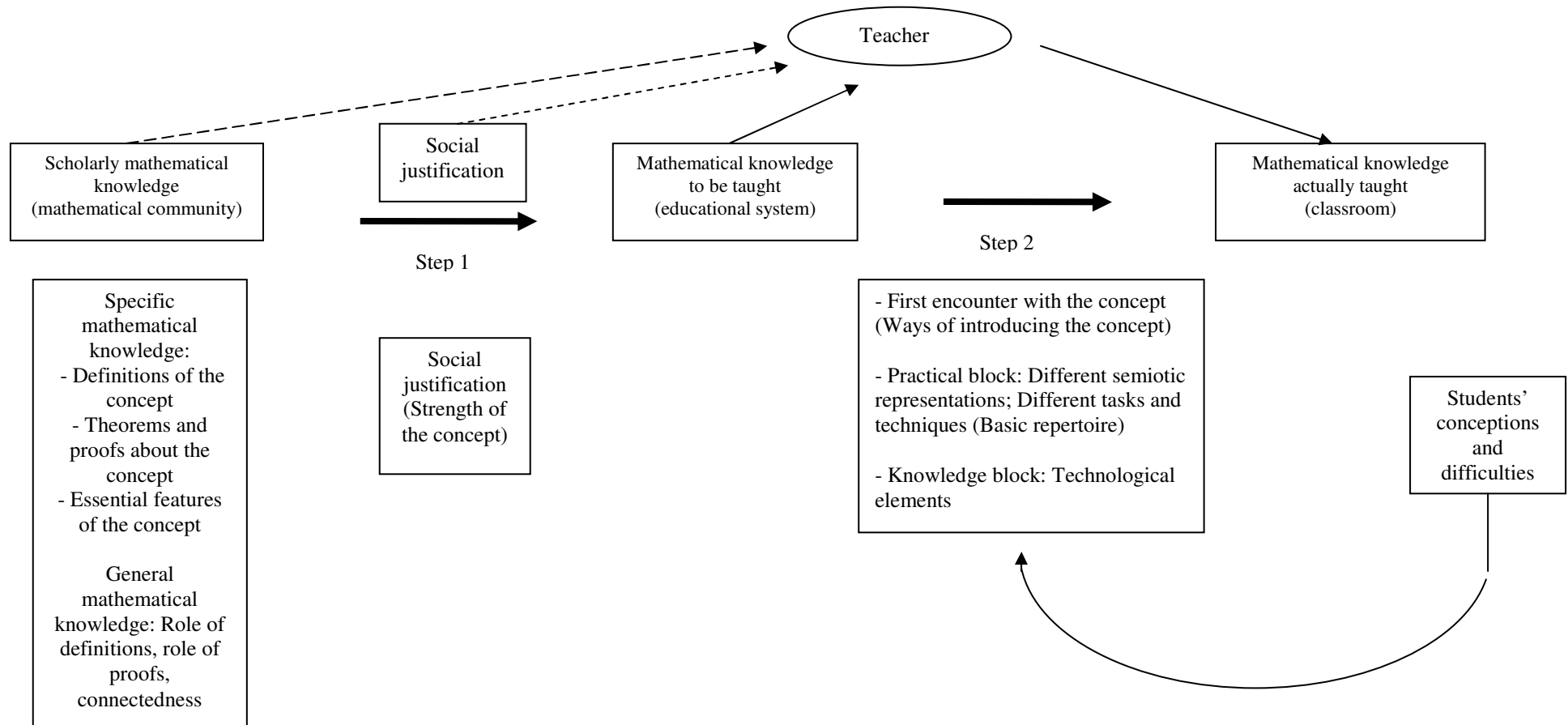


Figure 3.5 Mathematics for teaching and the didactical transposition

# **CHAPTER 4**

## **MATHEMATICS FOR TEACHING LIMITS OF FUNCTIONS**

## 4 Mathematics for Teaching Limits of Functions

In the last chapter, I developed a framework to analyse the knowledge needed by a mathematics teacher in order to teach a specific topic. As a starting point, I discussed the framework developed by Even (1990, 1993), based on the notions of Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) introduced by Shulman (1986, 1987). I then looked at this framework through the lens of Chevallard's theories, in particular the theory of didactical transposition (Chevallard, 1985, 1991) and the anthropological theory of didactics (Chevallard, 1999). This led me to classify *mathematics for teaching* (MfT) a specific concept according to the following categories: scholarly mathematical knowledge of the concept; knowledge about the social justification to teach this concept in a specific institution and at a specific level; knowledge about how to organise the students' first encounter with the concept; knowledge about the practical block (tasks and techniques); knowledge about the knowledge block; knowledge about students' conceptions and difficulties when studying this concept.

In this chapter I analyse *mathematics for teaching* limits of functions in the context of Mozambican secondary schools. This analysis is based on a review of literature in the field, conceptually organised according to my categories of MfT a specific concept.

Therefore, this chapter is structured as follows:

- 4.1. The scholarly mathematical knowledge
- 4.2. The social justification
- 4.3. The first encounter
- 4.4. The practical block
- 4.5. The construction of the knowledge block
- 4.6. Students' conceptions and difficulties
- 4.7. Conclusion

## 4.1 The scholarly mathematical knowledge

In order to teach a concept in schools, teachers first need to have a good knowledge of the scholarly mathematical knowledge on this concept, which is the starting point of the didactical transposition. An important part of the scholarly mathematical knowledge on limits of functions is the knowledge block of the reference MO, as presented in Chapter 2, which includes the formal definition of limits, theorems about limits, and their proofs. These can be found in many university level textbooks and I will not elaborate them here.

Another important part of this knowledge relates to the essential features of the concept, which emerge from its epistemological analysis. I develop this aspect in this section, considering the three main facets of the limit concept that have been underscored by several authors (Bkouche, 1996; Trouche, 1996):

- A dynamic point of view, related to the idea of movement: when a variable  $x$  tends to a value  $a$ , the variable  $y$ , which depends on  $x$ , approaches a value  $b$ ;
- A static point of view: for  $x$  more than a determined value, the distance between the  $y$ -values and the limit are less than a certain number. There is no idea of movement.
- An operational point of view: the limit works in accordance with rules.

I will show in the following sections that these three different features emerge from the history of the limit concept.

### 4.1.1. The dynamic point of view

The dynamic point of view was first developed by Isaac Newton (1642-1727) in the earliest definitions of limits of functions. Newton

considers mathematical quantities as generated 'by a continuous increase, in the same way as space is described by a moving object' and imagines 'the velocities of the movements that generate them'.

[considère les quantités mathématiques comme engendrées 'par une augmentation continue, à la manière de l'espace que décrit un corps en



mouvement' et imagine 'les vitesses des mouvements qui les engendrent'.]

(Dahan-Dalmenico & Peiffer, 1986: 192)<sup>7</sup>

He calls these velocities *fluxions*. Then he

tries to eliminate any trace of infinitely small magnitudes, firstly by considering only their ratio, and then by conceiving what will be his third method, the “method of first and ultimate ratios”.

[tente d'éliminer toute trace d'infiniment petit, d'abord en ne considérant que leur rapports, puis en concevant ce qui sera la troisième méthode, “la méthode des premières et dernières raisons”.] (1986: 193)

What Newton calls the *ultimate ratio of evanescent quantities* corresponds to the limit of their ratio and he defines it in the following terms:

By the ultimate ratio of evanescent quantities (i.e., ones that are approaching zero) is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. ... Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than any given difference, but never go beyond, nor in effect attain to, till quantities are diminished 'in infinitum'.

(Quoted by Edwards, who considers this definition as “his clearest exposition of the limit concept in which that calculus is based”, 1937: 225-226)

According to Edwards, the description [ $\lim_{x \rightarrow a} f(x) = L$  provides that  $f(x)$  approaches  $L$  as  $x$  approaches  $a$ ] used before Weiestrass is “tinged with connotations of continuous motion”. (1937: 333)

In fact, in all these attempts to define limits, the idea of movement is present: first through the idea of *moving object*, then through the notion of *evanescent quantities*, and finally in the idea of *approaching* something, which relate to the concept of function.

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<sup>7</sup> All quotations from French literature have been translated by myself

### 4.1.2. The static point of view

Augustin-Louis Cauchy (1789-1857) gave a definition, also related to the concept of function, reflecting a more static point of view.

When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, the last [fixed value] is called the ‘limit’ of all the others. Thus, for example, an irrational number is the limit of diverse fractions which furnish more and more approximate values of it. (quoted by Edwards, 1937: 310)

In fact, in this quote, both dynamic and static points of view are present. There is the idea of approaching (dynamic point of view) but also the idea of fixed values for the variable, close to the fixed value of the limit (static point of view).

Infinitely small quantities are seen as variables which tend to zero. This definition indicates an evolution towards a more static view.

According to Edwards,

the final loose end was tied by Weierstrass in his purely arithmetical formulation of the limit concept [...] it was said that  $\lim_{x \rightarrow a} f(x) = L$  provided that, given  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that  $|f(x) - L| < \varepsilon$  if  $0 < |x - a| < \delta$ . (1937: 333)

This modern definition, by Karl Weierstrass (1815-1897), is a static formulation involving only real numbers, without any idea of movement. In this definition, there is an inversion in the order of the variables  $x$  and  $y$ . While in the dynamic definition, when  $x$  approaches  $a$ ,  $f(x)$  approaches  $b$ , in this static definition the radius  $\varepsilon$  is chosen (arbitrarily small), although related to  $y$ , and  $\delta$  depends on  $\varepsilon$ . This makes this definition difficult to understand because students learn in schools that  $x$  is an independent variable, that can be chosen, and that  $y$  is the dependant variable, depending on the  $x$  previously chosen. The chosen radius is automatically linked with the independent variable.

This difficulty has been described by Courant and Robbins (1978), quoted by Fischbein (1993), in the case of limits of sequences.

There is a definite psychological difficulty in grasping this precise definition of limit. Our intuition suggests a “dynamic” idea of a limit as the result of the process

of “motion”: We move on through the row of integers  $1, 2, 3, \dots, n, \dots$  and then observe the behavior of the sequence  $a_n$ . We feel that the approach  $a_n \rightarrow a$  should be observable. But this “natural” attitude is not capable of clear mathematical formulation. To arrive at a precise definition we must *reverse* the order of steps; instead of looking at the independent variable  $n$  and then at the dependent variable  $a_n$ , we must base our definition on what we have to do if we wish actually to check the statement  $a_n \rightarrow a$ . In such a procedure, we must first choose an arbitrarily small margin around  $a$  and then determine whether we can meet this condition by taking the independent variable  $n$  sufficiently large. Then, by giving symbolic names,  $\varepsilon$  and  $N$ , to the phrases “arbitrarily small margin” and “sufficiently large  $n$ ” we are led to the precise definition of limits (1993: 238)

We will see that because of this inversion, even teachers have difficulties in understanding this definition.

More recently, topological definitions have been developed.

Let  $(a_i) (i = 1, 2, \dots, n, \dots)$  be a sequence of points of a space  $E$ . We say that this sequence **converges** to a point  $a$  of  $E$ , or that  $a$  is the **limit** of this sequence, if for every neighborhood  $V$  of  $a$  there exists an integer  $i_0$  such that  $a_i \in V$  for every  $i \geq i_0$ . (Choquet, 1966: 23)

Let  $f$  be a mapping of a set  $X$  into a topological space set  $Y$ ; let  $\mathcal{B}$  be a filter base on  $X$ , and let  $b$  be a point of  $Y$ . We say that  $f$  converges to  $b$  (or has limit  $b$ ) along  $\mathcal{B}$  if for every neighborhood  $V$  of  $b$  there exists a  $B \in \mathcal{B}$  such that  $f(B) \subset V$ . We then write  $\lim_{\mathcal{B}} f = b$ . (1966: 25)

These definitions continue with the static tradition.

The static definitions of limits of functions are the definitions that are actually used by the community of mathematicians. According to Robinet (1983), in an early stage the limit concept was developed to solve problems such as determining the slope of a tangent line, determining an asymptote, calculating indeterminate forms or the remainder of a series. At that stage, it was used implicitly. Nevertheless the formalisation of the concept does not come from the problems with which it originated, but was provoked

on the one hand, by the wish to validate the statements of the mathematicians who used the notion in an implicit way, and on the other hand to be able to demonstrate general theorems for entire classes of functions (and not only for one function given explicitly).

[par d'une part le désir de valider les affirmations des mathématiciens qui utilisaient la notion de manière implicite, d'autre part de pouvoir démontrer des théorèmes généraux pour des classes entières de fonctions (et pas pour une fonction donnée explicite).] (Robinet, 1983: 239)

The dynamic point of view was developed to solve problems, but the static point of view was necessary to formalise the limit concept.

### **4.1.3. The operational point of view**

According to Dahan-Dalmenico & Peiffer (1986), Leonard Euler (1707-1783) developed the study of limits using a formal point of view. He tried to clarify the rules instead of studying the nature of the objects involved in the operations. As I already showed, this is the point of view mainly developed in Mozambican secondary schools.

### **4.1.4. Overview**

The limit concept can be seen from three different points of view. The dynamic point of view was the first developed to solve problems, while the static point of view was developed to formalise the concept and the operational point of view to calculate limits. These three points of view are not opposed but complementary, as stated by Trouche (1996). Quoting Bkouche, he argues that in calculus two main aspects of mathematical thinking can be found: the intuitive thinking “too fuzzy for ensuring safe operations, both at the reasoning level and at the calculating level” [trop flou pour assurer une sécurité opératoire, autant sur le plan du raisonnement que sur le plan du calcul] and the formal thinking “which undertakes these safe operations”. [qui prend en charge cette sécurité opératoire] (Bkouche, quoted by Trouche, 1996: 81). The intuitive thinking corresponds to the dynamic point of view, the formal thinking to the static and operational points of view.

Trouche (1996) also asserts that, although the static point of view gives form to the concept presently accepted by the mathematician community, the study of textbooks and national examinations of French secondary school shows that at that time (1996) the operational point of view was dominant.

The study of the Mozambican didactic institutions' relation to the limit concept leads to the same conclusion (see Chapter 2). Two points of view are developed: the static point of view, through the formal definition, which is never used in practice, and the operational point of view, through the application of algebraic rules, which is dominant.

The formal definition taught in Mozambican secondary schools is usually the Weierstrass definition. Eight of the nine teachers interviewed in our previous study (Huillet & Mutemba, 2000) declared that they teach the  $\varepsilon$ - $\delta$  definition. Most of them are aware that students do not understand it and will not use it at secondary school level. One of the teachers interviewed said that even some teachers do not understand it. In fact, this definition is very abstract and may not help in developing a good concept image in an immediate sense, whereas a good concept image is necessary to understand the formal definition.

We also saw that in national examinations, the operational point of view was dominant. This form of examination has strong consequences for the formation of students' concept image of limits of functions.

Mutemba (2001) studied the concept image on limits of functions of Mozambican students through a questionnaire applied to 84 Grade 12 students, and interviews with 9 of them. From the analysis of students' answers, she concluded that 51 of them (61%) had a static image of this concept, while only 33 (39%) held a dynamic image. Of the students holding a dynamic concept image, most of them (28 out of 33) did not distinguish limit from asymptote and consequently, considered that a limit could never be reached. The other 5 students had a *motion picture* conception: as  $x$  goes to  $a$ , the corresponding values of the function approaches  $b$  indefinitely. Of the students holding a static concept image, 6 saw the limit as a *barrier*, 25 as a *value correspondence* (the limit is a  $y$ -value

corresponding to an  $x$ -value) and 20 as *procedural* (the limit concept is encapsulated in the mathematical procedures and rules).

Using the classification of three categories (dynamic, static and operational), the two last aspects by Mutemba (value correspondence and procedural) correspond to the “operational point of view”, the *value correspondence* being applied to a discontinuous function and the *procedural* to continuous functions, handling indeterminate forms. I would then classify her results as follows:

dynamic point of view	33	39%
static point of view	6	7%
operational point of view	45	54%

The operational point of view is predominant, as suggested by the study of the Mozambican secondary schools’ relation to this concept. We can surmise that these students have a very poor concept image of limits of functions, which does not reflect the richness of this concept. This concept image has been shaped by the kind of objects that they met in schools, which are usually monotonous convergent functions.

What concept image about limits of functions do Mozambican teachers hold? Are they aware of the different features of this concept? Do they have the same conceptions as observed in their students? Do they understand the  $\varepsilon$ - $\delta$  definition?

It is important for a mathematics teacher to understand the formal definition, because it is the way it has been formalised by mathematicians and it allows them to make general proofs. Moreover, if they deeply understand the definition, they will be able to see the difficulties in it, to consider whether a secondary school student would be able to understand it and why, and consequently to decide whether it is convenient to teach it in schools. On the other hand, it is also important that teachers have an intuitive knowledge of limits. For example they need to understand why a function increases (or decreases) quicker than another, in order to anticipate, to understand and to check the result of a calculation. This intuitive knowledge relates to a dynamic point of view.

In conclusion I would say that, to deeply understand the limit concept, teachers need to be aware of the three different features of the limit concept, and be able to switch between them in a flexible way. They also need to know and understand the  $\varepsilon$ - $\delta$  definition, as well as less formal definitions, and be able to choose a working definition understandable to their students. They also need to know the main theorems about limits and their proofs, to be able to correctly use the limit symbol and to know how to use it in association with other mathematical notation. All these aspects constitute the knowledge block of the reference MO about limits of functions.

## 4.2 The social justification

The first step of the didactical transposition consists of identifying objects that ought to be taught in schools. A social justification<sup>8</sup> must legitimate these choices. In Even's words, this relates to the strength of the concept.

What legitimates the teaching of limits of functions in Mozambican secondary schools?

In the first place, the concept of limits of functions has strong links with other fundamental mathematical concepts. It is built on the concepts of function and infinity, and it is also the basic concept for differential and integral calculus. The epistemological study of this concept shows that it played a fundamental role in defining the notions of derivatives and integrals, which are the bases of many other mathematical concepts, such as numerical series and series of functions. As stated by Tall, "although the function concept is central to modern mathematics, it is the concept of limit that signifies a move to a higher plane of mathematical thinking" (1992: 501).

Furthermore, the limit concept has many applications in other sciences. A well known application in physics is instant velocity. At secondary school level, we can also find applications in biology, for example to the study of a bacterial culture growth (Larson et al., 1994: 358) or of an epidemic spread (Hoffmann & Bradley,

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<sup>8</sup> As explained in Chapter 3, *social justification* means justification *for the society* and does not indicate the nature of the justification, which can be for example epistemological, political, professional, or social.

1996: 291), in economics, in particular with the logistic function (Huillet, 2000a: 71-72) and in other sciences.

The previous study of some Mozambican teachers' relation to limits of functions (Huillet & Mutemba, 2000) reveals that teachers do not understand why this concept is taught in schools. Some of them even consider the study of limits of functions in schools as an application of algebraic rules such as factorisation, cancellation, rationalisation, calculation with powers, and roots. It is important that teachers are aware of the key role that this concept plays in mathematics and in the learning of mathematics in schools, not only because it is the first abstract concept met by the students, but also because of all the possibilities of working on this concept using different representations and different applications.

### 4.3 The first encounter

When planning the didactical unit on limits of functions, teachers must organise their students' first encounter with this concept. This first encounter can have significant implications for students' concept image and, for this reason, must be carefully planned. This means that mathematics teachers need to know several ways of organising students' first encounter with limits, be able to analyse the influence of these different approaches on student's understanding and, as a consequence, be able to choose one or more of them for their lessons.

I will review here some of the different ways of organising the first encounter with the limit concept in schools.

Larson et al. (1994), for example, introduce limits through the “Tangent Line problem”. They show that as a point Q of a graph approaches the point P of the same graph, the slope of the secant line PQ approaches the slope of the tangent line. They conclude that “when such a ‘limiting position’ exists, the slope of the tangent line is said to be the **limit** of the slope of the secant line” (1994: 61). Then they change to the numerical register, using the values of the function

$f(x) = \frac{x^3 - 1}{x - 1}$  to show that  $f(x)$  approaches 3 when  $x$  approaches 1 from the left

and from the right (1994: 62). Using the numerical register, they introduce different possibilities of limits: behaviour that differs from the right and from the



left, unbounded behaviour, oscillating behaviour. The next step consists of introducing a formal definition, followed by some examples of how to use this definition. The chapter on limits continues with properties of limits, techniques for evaluating limits, continuity, and infinite limits. In most of the examples given in their book, there is a strong link between algebraic, graphical and numerical registers on the one hand, with the use of calculator and graphical utility on the other.

Some mathematics educators studied the teaching and learning of limits of functions and, using the results of their research, constructed didactical sequences for its teaching. I will refer here to the works related by Robinet (1983), Cornu (1984), Sierpinska (1987) and Schneider (2001).

Robinet contends that, to introduce limits of functions at secondary school, there are two basic types of approaches:

- An entirely qualitative approach which does not allow us to establish general theorems, but which links the notion of limits with the real phenomena which give rise to it;
  - An entirely formalised approach which allows us to solve limit problems for non explicit functions, but can provoke formal mismatching.
- [ - Une approche complètement qualitative qui ne permettrait pas d'établir de théorèmes généraux, mais qui lierait bien la notion de limites aux phénomènes réels qui peuvent lui donner du sens.
- Une approche complètement formalisée qui permet de résoudre des problèmes de limites pour des fonctions non explicitées, mais qui risque d'occasionner des décalages formels.] (1983: 286)

She suggests that it is difficult to find the right place between these two extremes.

Using a midway solution, she elaborated a teaching sequence for the notion of limits of function, considering its place inside the teaching of mathematics, and the learners' age. To choose a problematic situation for the introduction of limits, she considered several possibilities:

- A problem related to derivatives, supported by the tangent notion, which was already familiar to students;

- Starting with continuity, as many French textbooks do, asking students to sketch the graph of several continuous and non-continuous functions, and to classify them;
- Studying the behaviour of functions which tend to plus infinity or minus infinity as  $x$  tends to plus or minus infinity.

The third possibility was used for the teaching sequence. She concludes that, as expected, it is difficult for a secondary school student to make proper use of the formal definition, and that

the major problem for the teaching of the notion of limit is to decide which knowledge and which know-how we want the students to acquire.

[le problème majeur dans l'enseignement de la notion de limite est de décider quels sont les savoirs et savoir-faire que l'on veut faire acquérir aux élèves.] (Robinet, 1983: 286)

For his experiment, Cornu (1984) divided a class in three groups. Each group had to solve a task linked to the limit concept in different settings: calculating the ratio between the areas of two circles, finding the slope of a tangent line, and a task about the development of decimal numbers. After a period of individual work, the students had to exchange their results. They were expected to establish a link between the three activities, showing by evidence the underlying common notion. However, the students did not seem to perceive the idea of limit present in all three tasks.

Sierpinska (1987) elaborated didactical situations aiming to help students overcome epistemological obstacles related to limits. She chose infinite series as mathematical context, and worked with humanities students on four 45 minutes sessions. She concluded that the epistemological obstacles which she had previously identified had not been completely overcome by any of the students, but that mental conflicts arose, that could be a starting point to overcome these difficulties.

More recently, the AHA Group (Approche Heuristique de l'Analyse) [Heuristic Approach of Calculus] conducted a teaching experiment on limits in an upper secondary school in Belgium (Schneider, 2001), based at the same time on

Brousseau's theory of didactic situations (Brousseau, 1998) and Chevallard's anthropological theory of didactics (Chevallard, 1992 and 1999). In this work, the first approach was made through the tangent line problem, but without speaking explicitly of limit. Another approach was made through problems of velocity, and then the link made between instantaneous velocity and slopes of tangent lines. All the students' work was based on "discoveries" made by solving problems. The observation of classes working through this method led the AHA Group to ask several questions about the part given to students and that controlled by the teacher in the construction of a new knowledge.

All these research reports show that the introduction of the limit concept in schools is seen as problematic by many mathematics educators and in several countries. On the other hand, the results of these experiments give us an idea about the difficulties that students face in understanding this concept.

As already explained (see Chapter 2), in Mozambique the limit concept is usually introduced through sequences, with some numerical and graphical interpretation. Some teachers introduce the  $\varepsilon$ - $\delta$  definition, which is never used in practice. Most of them quickly turn to algebraic tasks.

Teachers do not seem to question the way limits are usually introduced at secondary school level. It would be important for them to reflect on this first encounter with limits, to have access to other approaches and to analyse the consequences they can have for their students' understanding of this concept. I indicate here some possible introductions to the concept, which have already been used in textbooks or by mathematics education researchers. This list is not exhaustive, and the epistemological study of the limit concept, as well as its study in different registers and its applications, can suggest other possibilities.

#### **4.4 The practical block**

In addition to the organisation of their students' first encounter with the limit concept, teachers must give them some tasks and introduce some techniques to solve these tasks. In Mozambican secondary schools, most of the tasks are algebraic, which lead to indeterminate forms. What other kinds of tasks could be used at this level in order to develop a deeper understanding of the limit concept?

As explained in Chapter 3, a mathematical object can be represented using different settings (epistemological point of view) and different semiotic registers (cognitive point of view). Changes of settings, or shifts from one register to another, allow the learner to access new information and, consequently, to construct a deeper understanding of this object. Therefore, teachers should be able to give their students tasks in, or leading to, different representations<sup>9</sup>. In Even's terms, they need to broaden their basic repertoire.

In this section I analyse the different settings and registers where the limit concept can be studied, the possible changes of register, and which new kind of tasks could compose Mozambican teachers' basic repertoire.

#### **4.4.1 Epistemological point of view: different settings**

The history of the limit concept shows (Dahan-Dalmenico & Peiffer, 1986) that it has been studied in different settings in different times. Its development is linked to the concepts of *infinity* and *of infinitely large and small magnitudes*. I will survey several possibilities of studying the limit concept in different settings, considering the following settings: geometrical, numerical, formal, algebraic and topological.

##### ***Geometrical setting***

The first approximation to limits made by the Greeks takes place in the geometrical setting. It is related to problems of determining areas and volumes. "The area of a circle can be approximated arbitrarily closely by the area of an inscribed regular polygon with sufficiently many sides." (Edwards, 1937: 7) The study of the area of the circle as the limit of subscribed and circumscribed polygons continues during the development of the limit concept at different times (Archimedes, Stevin, Pascal).

Other problems belonging to the geometrical setting have been relevant to the development of the limit concept. They are the division of a segment leading to the infinite series  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ , the determination of lengths of curves and

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<sup>9</sup> I use the term *representation* as a broader term to indicate both settings and registers

areas of figures (Kepler, Cavalieri, Torricelli, Fermat), and the calculation of volumes of solids (Kepler, Cavalieri, Torricelli, Fermat). Tangent line considered as limit of a secant (Fermat) also belongs to this setting.

Johan Kepler (1571-1630), for example, breaks up areas and volumes into an infinite number of infinitesimal pieces of the same dimension, called “*indivisibles*”. In *Nova Stereometria Doliorum Vinariorum* (New Solid Geometry of Wine Barrels), published in 1615, he uses this procedure to gauge the volumes of wine barrels.

Bonaventura Cavalieri (1598-1647) regards

an area as consisting of parallel and equidistant line segments, and a volume as consisting of parallel and equidistant plane sections, without making entirely clear whether these indivisible units have thickness or not.

(Edwards, 1937: 104)

He does not speculate about the nature of the infinity, but avoids calculating an area as the sum of its indivisible units. Instead he determines the ratio between areas of which indivisible components are in a constant ratio (Dahan-Dalmenico & Peiffer, 1986: 179). Evangelista Torricelli (1608-1647) uses this method considering cylindrical indivisibles instead of plane ones.

In Mozambican secondary schools, the limit concept appears implicitly for the first time in Grade 7, when studying the area of a circle. This example could be used as a starting point for the formal study of limits in Grade 12. Some other simple tasks could also be used, as for example the division of a segment presented above.

### ***Numerical setting***

In this setting we find the study of infinite series such as  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ , linked to the geometrical setting (Zenon, Stevin, Valero, Robertval, Fermat, Pascal), and the definition of a real number as the limit of a sequence of rational numbers called “fundamental sequence” (Cauchy).

Trouche (1996) includes the current  $\varepsilon$ - $\delta$  definition in the numerical setting. In fact it marked the historical return to the numerical setting made by Weierstrass. Nevertheless, given the importance of this formal definition in the development of the limit concept, and that it has given rise to very specific tasks in schools or at university, I will consider the formal setting separately.

### ***Formal setting***

The formal definition of the limit concept was developed by Karl Weierstrass (1815-1897). It is a static formulation involving only real numbers.

$\lim_{x \rightarrow a} f(x) = L$  provides that, given  $\varepsilon > 0$  there exists a number  $\delta > 0$  such that if  $|f(x) - L| < \varepsilon$  then  $0 < |x - a| < \delta$ . (Edwards, 1937: 333)

The limit concept can be studied in the formal setting, for example the task:

Prove that  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$ .

This task belongs to the formal setting, and the objects manipulated to solve it are different from the following task, which belongs to the algebraic setting:

Determine  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ .

Given the difficulties inherent in the formal definition, tasks in the formal setting are not suitable at secondary school level. Nevertheless, it is important for mathematics teachers to understand them.

### ***Algebraic setting***

The development of rules for calculations with infinitely small magnitudes, made by Newton and Leibniz, moved the study of the limit concept from geometrical and numerical settings to the algebraic setting. It is in this setting that most of the work about limits of functions is done at secondary school level in Mozambique. However, even this setting is sometimes used in a much standardised way, which does not allow students to access new information (Huillet, 2000b). Using a specific function, this situation is exemplified below.

In the algebraic setting, some functions can be represented in several ways, as This is the case of many rational functions. For example, the function defined by

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 7x + 12} \quad (1)$$

can be represented as

$$f(x) = \frac{(x-1)(x-2)}{(x-3)(x-4)} \quad (2) \text{ by factorisation,}$$

$$f(x) = 1 + \frac{4x-10}{x^2-7x+12} \quad (3) \text{ by division,}$$

$$f(x) = 1 + \frac{4x-10}{(x-3)(x-4)} \quad (4) \text{ by division and factorisation,}$$

$$\text{or } f(x) = 1 - \frac{2}{x-3} + \frac{6}{x-4} \quad (5) \text{ by decomposing the fraction into simple elements.}$$

Each algebraic representation of the function plays a different role in calculating the limits of this function. For example the models (2), (4) and (5) can facilitate the calculation of the limits when  $x$  goes to 3 and when  $x$  goes to 4, both from the left and from the right. The models (3) and (5) are more useful to determine the limit when  $x$  goes to  $+\infty$  or  $-\infty$ . I will expand on this point later, in “changes of representations”.

### ***Topological setting***

In its more recent development, the limit concept has been studied in the topological setting. Even if this setting is not appropriate for secondary school level, it may be important for teachers to have knowledge of it. The concept of *neighbourhood* is not as abstract to students as the  $\varepsilon$ - $\delta$  concept is and could be used in some teachers' explanations.

#### **4.4.2 Cognitive point of view: different registers**

I will now look at the use of different semiotic registers in the study of the limit concept, in particular the numerical, the linguistic, and the graphical registers.

##### ***Numerical register***

Depending on the point of view (epistemological or cognitive) we can consider a numerical representation as a setting or as a register.

We already saw that the task “Find the sum of the series  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ ”

belongs to the numerical setting. The mathematical object involved in the task, a series, is numerical. The procedures used to solve this task are also specific to this kind of mathematical object.

The task “Determine the limit  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ ” is an algebraic task but, to get an intuitive result of this limit, we can use numerical values, as shown in the following table.

$x$	1.9	1.99	1.999	2	2.001	2.01	2.1
$\frac{x^2 - 4}{x - 2}$	3.9	3.99	3.999	XXX	4.001	4.01	4.1

The observation of this table shows that there is regularity in the results of the calculations. This leads to the conjecture: the limit must be 4.

In this case, the numerical register has been used as a cognitive tool that helps us give meaning to this task. The problem remains the same, as stated in the algebraic setting, and no new mathematical object has been created, yet we changed the semiotic register.

In Mozambique, this kind of numerical representation is sometimes used by teachers when introducing the limit concept. However, no further task of this kind is given to students. I argue that an expanded use of the numerical register, especially at the beginning of the work with limits, could help students give meaning to this concept.

### ***Linguistic register***

The linguistic register has been the focus of several authors, as it plays an important role in concept formation in mathematics.

Laborde asserts that we should

acknowledge the existence of specific conceptual problems set by language activity either of formulation or of understanding (...) and look at language problems as interconnected with problems of knowledge construction.



[reconnaître l'existence de problèmes conceptuels spécifiques posés par l'activité langagière qu'elle soit de formulation ou de compréhension (...) et envisager les problèmes langagiers en interdépendance avec ceux de la construction des connaissances.] (1992 : 8)

In fact, in a mathematical discourse, we find

two codes, the natural language and what we will call the symbolic writing, constituted by external symbols subjected to specific ordering rules between symbols.

[deux codes, la langue naturelle et ce que nous appelons l'écriture symbolique, formée de signes extérieurs soumise à des règles spécifiques d'agencement entre signes.] (1992: 12)

Laborde (1992) adds that

the language used in mathematics is not a simple juxtaposition of these two codes but a language which results from a real interaction between these codes.

[la langue utilisée en mathématiques n'est pas la simple juxtaposition de ces deux codes mais une langue résultat d'une véritable interaction entre ces deux codes.] (1992: 15)

Several authors addressed students' language issues related to limits of functions.

I will consider three main linguistic problems. These are:

- the way limits are spoken about, using natural language,
- the use of symbols when working with limits, and
- the way the limit symbol is read in natural language.

Cornu (1983, 1991) analyses French students' spontaneous conceptions of limits in connection with the influence of language. He observes that the words "*tends to*" and "*limit*" have significance for students before any lesson about the limit concept and that they continue to rely on these meanings afterward. He writes:

Research has revealed many different meanings for the expression 'tends toward':

- to approach (eventually staying away from it)
- to approach ... without reaching it
- to approach ... just reaching it

- to resemble (without any variation, such as 'this blue tends towards violet') (Cornu, 1991: 154).

The word "limit" can be seen as:

- an impassable limit which is reachable,
- an impassable limit which is impossible to reach,
- a point which one approaches, without reaching it,
- a point which one approaches and reaches,
- a higher (or lower) limit,
- a maximum or minimum
- an interval,
- that which comes 'immediately after' what can be attained,
- a constraint, a ban, a rule,
- the end, the finish. (1991: 154-155)

Monaghan studied the ambiguities inherent in the four expressions “*tends to*”, “*approaches*”, “*converges*” and “*limit*” for English students. He concludes that, although these expressions are interchangeable for the mathematician, for the student they are not.

‘Approaches’ appears to present the least difficulties to students because it is a vague term. ‘Tends to’ is often seen as similar in meaning to ‘approaches’ in mathematical contexts although its everyday use does not suggest limit situations. Both phrases are given a dynamic interpretation. ‘Converges’ is confusing in that its everyday meaning is strongly associated with lines converging. (...) Limit is often viewed as a boundary point. (1991: 23)

In Mozambique, the official and teaching language is Portuguese, but it is not the mother tongue of most of the students, who usually speak an African language at home and learn Portuguese at school. Many teachers may also be influenced by their mother tongue as well as by the Portuguese natural language when learning and teaching mathematics in general, and particularly the limit concept. This

special linguistic situation might have a specific influence on the learning of the limit concept in the Mozambican context.<sup>10</sup>

Moreover, students usually have difficulties with logical symbols, even at university. This is the case for example with the quantifiers  $\forall$  and  $\exists$  that they often substitute for the ‘corresponding’ Portuguese expressions. For instance, for solving an equation such as  $x^2 + 1 = 0$ , they would write “ $x = \nexists$ ” ( $x$  does not exist), meaning that the equation does not have any real solution (there does not exist any  $x \in \mathbb{R}$  such that  $x^2 + 1 = 0$ ). Writing the symbolic expression  $\lim_{x \rightarrow a} f(x) = b$  is also difficult for some students to write. They often make it  $\lim_{x \rightarrow a} f(x) = b$ , or even  $\lim_{x \rightarrow a} f(x) = b$ , extending  $x \rightarrow a$  under the second term of the equality.

Furthermore, the way many students read the symbolic expression  $\lim_{x \rightarrow a} f(x) = b$  may also be a consequence of their linguistic difficulties. The correct way of reading the expression  $\lim_{x \rightarrow a} f(x) = b$  in Portuguese would be “limite de  $f$  de  $x$  quando  $x$  tende para  $a$  é igual a  $b$ ” [limit of  $f$  of  $x$  when  $x$  goes to  $a$ , equals  $b$ ]. Many students read it as “limite de  $x$  quando tende para  $a$  é igual a  $b$ ” [limit of  $x$  when it goes to  $a$ , equals  $b$ ]. This reading misses the meaning of the expression, because it speaks about the limit of  $x$  instead of the limit of  $f(x)$ . It seems that these students just memorised an expression that they do not understand.

This incorrect use of language and symbols when working with limits clearly shows students’ difficulties in giving meaning to the limit concept. If not corrected and explained, it can deepen their difficulties in understanding this concept. An important role of mathematics teachers is helping students to use correctly mathematical language and symbols.

### ***Graphical register***

Graphs play a special role and have a special status in mathematics. On the one hand, they are very useful to visualize and help to form conjectures when solving

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<sup>10</sup> In fact this linguistic situation is not a specific part of the research, as it did not emerge during data collection and analysis.

some tasks. On the other hand they lack precision and can lead to incorrect results. For these reasons, the graphical register is not operational. We cannot use a graph to prove a theorem, because a graph represents a special case and a proof must be general. Nevertheless a graph can be very useful to help us understand a situation. This is also the case with limits. We already saw an example of how to use the graphical register to give meaning to the limit  $\lim_{x \rightarrow +\infty} (x - \ln x)$  (see page 72).

Furthermore, the limits of a function can be used to sketch its graph. This means that students should be able to link the result of a limit calculation to its graphical representation. This can be done in several ways, as I will show in the next section.

### 4.4.3 Changes of representation

We saw that the limit concept can be studied in different settings or using several registers. All these different aspects of limits of functions are complementary and changes of registers should help learners to reach a deeper understanding of this concept and more flexibility in their knowledge. For this reason it is important that teachers be aware of the several registers limits of functions can have in different settings. They also need to be able to “translate” limits from one representation to another.

The study of Mozambican secondary school relation to limits of functions shows that the only representations really used to handle limits in schools are, in the first place, the algebraic one and, with less importance, the graphical register, essentially used to read the limits from the graph in some final examinations.

However, there are many possible changes of representations. If we focus on the four main representations already identified for the learning of limits of functions in Mozambican secondary schools, we can represent these changes of representations in the following diagram (Figure 4.1, next page).

In this diagram I indicate the formal setting, because it is part of the Mozambican syllabus, even if I personally consider that it is not suitable for this level.

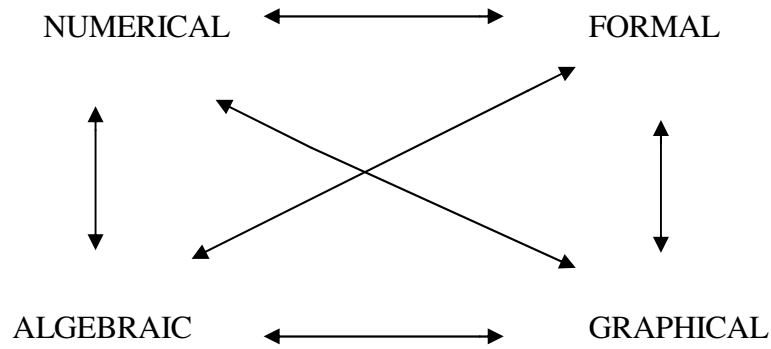


Figure 4.1 Changes of representations

In fact, from twelve possibilities of translating limits of functions from one representation to another, only one appears in the national exam papers: reading limits from a graph. Other tasks about limits are merely algebraic ones.

Even within the algebraic setting, which is dominant in Mozambican secondary schools, some algebraic representations of rational functions used to calculate limits do not allow a graphical interpretation (Huillet, 2000b). I will explain this

statement using the same rational function as before:  $f(x) = \frac{x^2 - 3x + 2}{x^2 - 7x + 12}$ .

In order to determine the limit of this function when  $x$  goes to  $+\infty$ , the following technique is usually taught in Mozambican secondary schools:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2 - 3x + 2}{x^2 - 7x + 12} = \lim_{x \rightarrow \infty} \frac{x^2 \left( 1 - \frac{3}{x} + \frac{2}{x^2} \right)}{x^2 \left( 1 - \frac{7}{x} + \frac{12}{x^2} \right)} = \lim_{x \rightarrow \infty} \frac{1 - \frac{3}{x} + \frac{2}{x^2}}{1 - \frac{7}{x} + \frac{12}{x^2}} = 1.$$

This new algebraic representation of the function,  $f(x) = \frac{1 - \frac{3}{x} + \frac{2}{x^2}}{1 - \frac{7}{x} + \frac{12}{x^2}}$ , does not

allow us to know whether  $f(x)$  goes to 1 from below or from above. This information would be necessary to represent this result on a graph. For this reason, I do not consider it as a “good” representation to evaluate this limit in order to sketch the graph of the function. The algebraic expression

$f(x) = 1 + \frac{4x - 10}{x^2 - 7x + 12}$  gives better information, showing that  $f(x)$  goes to 1 from

above when  $x \rightarrow +\infty$ , because in this case  $\frac{4x-10}{x^2-7x+12} > 0$ , and to 1 from below

when  $x \rightarrow -\infty$ , because in that case  $\frac{4x-10}{x^2-7x+12} < 0$ .

These results can easily be translated graphically, for example in the following way (Figure 4.2).

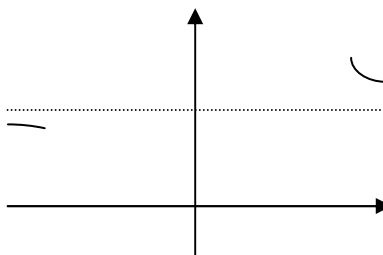


Figure 4.2 Graphical interpretation of a limit

Indicating limits on a graph is an important kind of task to prepare students to sketch graphs of functions. This kind of task does not occur in Mozambican schools. As a consequence, when sketching the graph of a function students are not able to use its limits.

In another situation, it may be irrelevant to know whether the function tends to its limit from below or from above. In that case, the technique usually taught in schools can be used. This means that different techniques can be used to evaluate the same limit, depending on the purpose of the calculation. A teacher should be able to help students develop flexibility in the use of algebraic techniques to evaluate limits. This means that his/her own knowledge needs to be flexible.

Another example of the use of different representations of the same task is called in Mozambique the “complete study of a function”<sup>11</sup>. From my point of view, based on my experience as a secondary school student in France, the aim of this complete study is to gather information about a function, starting from its analytical expression, in order to sketch its graph. It usually includes the study of the domain, symmetry tests and periodicity in order to reduce the interval of study, limits and asymptotes, first derivative and intervals of increase and decrease, sometimes second derivative and concavity. All this information allows

<sup>11</sup> In Mozambique, the “complete study” of a function includes finding the domain, limits, first derivative, intervals of increase and decrease, and sketching the graph.

us to sketch the graph. When performing this task, we should go back and forth between the algebraic and the graphical registers, interpreting the result of a calculation in the graph, and anticipating and checking the result of a calculation with the graph. In Mozambique, the complete study of a function is often done in a very formal way. Many students who enter university determine the domain, the limits and the derivative, but in any order, and without drawing any conclusion from the results. They can even have contradictory results, such as a function increasing while its limit is minus infinity. They then sketch the graph using several numerical values. As a consequence of this method, the graph can be very different from the result of the study, without the student noticing this incoherence. It seems that they do not think about mathematical work as something logical and coherent but as a set of techniques that they use when they are asked to, without any logic and usefulness. This can be illustrated by the following anecdote. One of my first year university students came to my office and asked me a question about how to solve a task. Instead of giving her the answer, I began asking her questions for her to reach the conclusion. When she got there she exclaimed: “It’s incredible. For you everything has an explanation!” For me this statement highlights the way students usually see mathematics in Mozambique, explanation not being part of the game. Do secondary school teachers have the same conception? We can suppose that some of them do and transmit this idea to their students.

The example of the complete study of a function shows evidence that the teaching of this topic fails in terms of connectedness. Students are required to perform isolated tasks within this study, but without linking them. Working with different representations could help students develop this connectedness.

#### **4.4.4 Basic repertoire**

Considering the possibilities of studying the limit concept in different settings and registers, and applying it in other sciences, it is obvious that the basic repertoire of Mozambican mathematics teachers, almost limited to algebraic tasks, needs to be broadened. In this section I suggest some kinds of tasks that could be integrated

into this repertoire. Obviously this list is not exhaustive, and new kinds of tasks could be introduced by teachers, depending on their students' difficulties.

### ***Numerical tasks***

In the numerical register, the students could use numerical values in order to approach a limit and understand what it means that the limit of a function has a finite value or is infinite. This kind of task could be used for students to develop an intuitive idea of basic limits such as  $\lim_{x \rightarrow \infty} P(x)$ , where  $P(x)$  is a polynomial,

$\lim_{x \rightarrow \infty} \frac{1}{x}$ ,  $\lim_{x \rightarrow 0} \frac{1}{x}$ ,  $\lim_{x \rightarrow \infty} \frac{ax+b}{cx+d}$ , limits of logarithmic and exponential functions, or special

limits such as  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$  and  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ . This kind of task could also be used to

compare the way different functions increase or decrease and, in that way, anticipate the result of an indeterminate form. Moreover, students should know that they can turn to the numerical register to anticipate a limit whenever they need to get an intuitive idea of this limit, as shown in Chapter 3 for the limit  $\lim_{x \rightarrow +\infty} (x - \ln x)$ .

### ***Graphical tasks***

The graphical register could be used in several ways. In the first place, students could use graphs to get an intuitive idea of a limit, as shown in Chapter 3 for the limit  $\lim_{x \rightarrow +\infty} (x - \ln x)$ . This means that they should be able to read limits from a graph, and to compare the way several functions increase or decrease. Then they should be able to give a graphical interpretation of a limit. Therefore, I suggest the following kinds of tasks:

- Reading limits from a given graph;
- Sketching the graph of two simple functions in order to compare their limits;
- Matching some graphs with their analytical expression, using the limits of the functions;
- Sketching a graph using only given information about limits;
- Using the analytical expression of a function, evaluate its limits and sketch its graph.



I suggest that these kinds of tasks would help students build a meaningful understanding of the limit concept.

### ***Algebraic tasks***

In Mozambican secondary schools, the limit concept is mainly handled in an algebraic setting, in a very standardized way. Instead of using standard techniques to evaluate each kind of limit, students should be encouraged to choose the more suitable analytical expression, depending on the one hand on the limit to be evaluated and, on the other hand, what they will use the result for (see Change of representation).

### ***Word problems***

I already mentioned (see Section 4.2) that the limit concept could be applied in other areas of mathematics, such as geometry, and in other sciences such as physics, biology, and economics. Some simple word problems using the limit concept would help students understand the usefulness of limits.

I have used some of these word problems at Eduardo Mondlane University in first year Biology and Veterinary courses, in order to lead students to a better understanding of the limit concept. My personal experience of these courses is that students face many difficulties in analysing and solving word problems, because they are not used to it.

Do teachers face the same difficulties? Do they ever solve tasks about limits different from the tasks usually solved at secondary school or in calculus courses? Teachers should have a basic repertoire of various different tasks about limits, and should be able to use them in their classes in order to help their students build a deep and broad understanding of the concept.

## **4.5 The construction of the knowledge block**

Besides the introduction of the concept and the practical block, the teacher must give some theoretical elements to his/her students to justify the techniques used to solve the tasks. This moment is not necessarily separated from the other moments of study, but can be integrated with them, depending on the kind of didactical organisation. For example, in traditional classes, the teacher usually introduces the

theoretical block, and then students are required to solve some tasks. In a more student-centred lesson, students can be given a task leading to the concept, and the theoretical block is introduced afterwards. I discuss here the mathematical knowledge needed by teachers to choose a theoretical block to teach the limit concept in schools, without taking into account the kind of didactical organisation chosen by this teacher.

As I already discussed in the analysis of the mathematical organisation (see Chapter 2), the nature of the limit concept and the inherent difficulties for students to understand the theoretical block of the reference mathematical organisation (MO<sub>2</sub>) restrain the possible didactical transposition at secondary school level. What kind of technological elements could a teacher introduce in a Mozambican secondary school? I will consider here three main aspects: the definition, the justification of the techniques, and the symbolic notation of a limit.

One important aspect of a concept is its definition. We already saw that the  $\varepsilon$ - $\delta$  definition of limits is complex and does not give a direct access to this concept. Therefore teachers need to reflect on the dilemma of teaching this definition at secondary school or not. This means that they must know other definitions that they could choose as an alternative to the  $\varepsilon$ - $\delta$  definition.

In our previous study (Huillet & Mutemba, 2000) some teachers said that they teach the  $\varepsilon$ - $\delta$  definition at secondary school, even knowing that students will not understand it. They argue that, in mathematics, a definition must be given for each concept. Can we say that these teachers understand the role of definitions in mathematics when they teach a definition being aware that their students will not understand it? Or do they have a formal idea that “a definition must be taught”, but without understanding exactly why? In my opinion it is not worth teaching this definition in secondary schools, because students will memorise it without understanding it. An alternative to this formal definition could be the dynamic definition:  $\lim_{x \rightarrow a} f(x) = L$  provides that  $f(x)$  approaches  $L$  as  $x$  approaches  $a$ . Even if this definition is not the definition recognized by the mathematicians’ community, it could be used in secondary schools to give students an intuitive idea of limits.

At some point the teacher must introduce the mathematical notation of limits.

Notation and symbols are often introduced in schools without explanation of their meaning, of their use, and of how to read them. We already saw some examples of this fact with the limit notation and the use of quantifiers (see Linguistic register).

Another example is the use of brackets. Many first year university students omit brackets. For example they would write  $x^2 + 1.x^3 + 2$  instead of  $(x^2 + 1)(x^3 + 2)$ .

They usually know that they mean the product of the two polynomial expressions, but in a more complex expression they can forget it over the calculation and get a wrong result. It does not seem that their secondary school teachers drew their attention to this fact. Are teachers aware of the importance of using rigorous mathematical notations and symbols? Do teachers themselves use notation appropriately? Teachers need to reflect on the symbolic notation of limits, on how to read it, and on students' difficulties with new symbolic notations.

Then the teacher must justify the rules used to calculate limits. The proofs of these rules are based on the  $\varepsilon$ - $\delta$  definition. If the teacher decides not to teach this formal definition, how could s/he justify the rules? S/he faces here a new dilemma. As a consequence, most of the rules about limits are given to students without a proof, as if they were transparent rules. Other possibilities to justify the rules could be to use different registers, for example the numerical and the graphical register, in order to verify these rules in some particular cases. Of course this would not constitute a proof, but at least students could understand why these rules work, at their own level.

As for the techniques to calculate limits, they are often taught without explanation, and without verifying that they work in some cases and not in other cases. As a consequence, students sometimes use a technique outside its validity domain. In the following example, a technique for  $x$  tending to infinity was used when  $x$  tends to a finite value.

$$\lim_{x \rightarrow 1} \frac{2x^2 + 3x - 5}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{x^2 \left( 2 + \frac{3}{x} - \frac{5}{x^2} \right)}{x^2 \left( 1 - \frac{1}{x^2} \right)} = \lim_{x \rightarrow 1} \frac{2 + \frac{3}{x} - \frac{5}{x^2}}{1 - \frac{1}{x^2}} = \frac{2}{1} = 2.$$

This obviously leads to a wrong answer. In this case the limit could be evaluated as follows.

$$\lim_{x \rightarrow 1} \frac{2x^2 + 3x - 5}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(2x+5)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{2x+5}{x+1} = \frac{7}{2}.$$

This example shows that some students do not understand the technical justification that explains why a technique works and what is its domain of validity. Teachers must reflect on how to justify each technique, depending on their students' difficulties.

To conclude, I would say that the choice of a theoretical block to teach limits in schools is not an easy task. It requires from the teacher a good knowledge of the theoretical block of the reference mathematical organisation (MO<sub>2</sub>), but also a deep understanding of students' previous knowledge and difficulties.

#### **4.6 Students conceptions and difficulties**

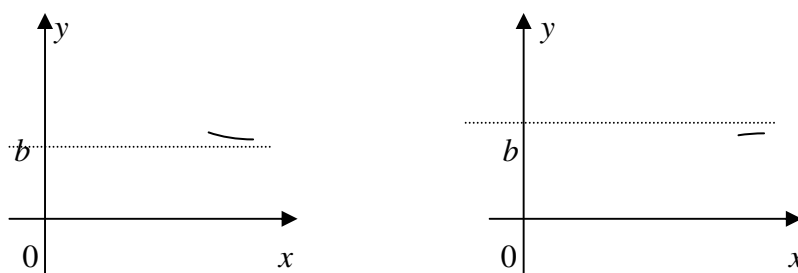
When teaching a mathematical topic, it is important that the teacher be aware of the different conceptions, and even “misconceptions” or “alternative conceptions”, held by students. It is also important that they have a good understanding of the difficulties that students face when learning the topic, and of the possible reasons for these difficulties. This knowledge should inform all the didactical transposition made by the teacher.

Ball et al. contend that “teaching involves more than recognizing that this student's answer is wrong. Teaching also entails analyzing the site and source of the error”. (2004: 52)

Some students' conceptions and difficulties when learning the limit concept already stood out when analysing other aspects of *mathematics for teaching* limits of functions. For example we saw that, according to Mutemba (2001), many students in Mozambique held an operational point of view. The same study revealed the poor concept image of limits of functions held by the students.

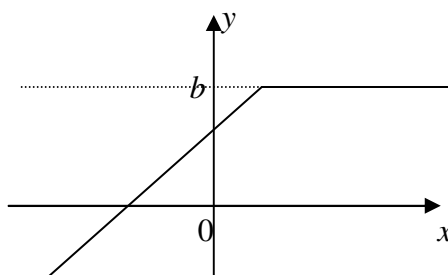
A poor concept image about limits of functions can lead students to misconceptions. For instance, examples given in schools of limits such as  $\lim_{x \rightarrow \infty} f(x) = b$  are generally those of rational functions, or exponential functions,

where the graph of the function approaches the asymptote  $y = b$ , with  $y > b$  or  $y < b$  (Figure 4.3).



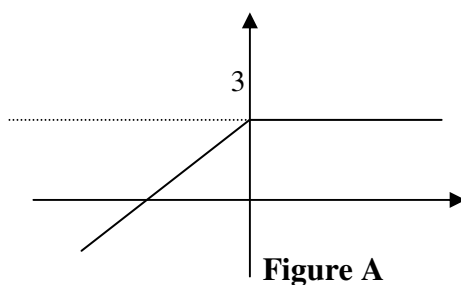
**Figure 4.3 Limits along a horizontal asymptote**

Many students have a strong idea that the limit of a function cannot be reached, as it is in the following example.

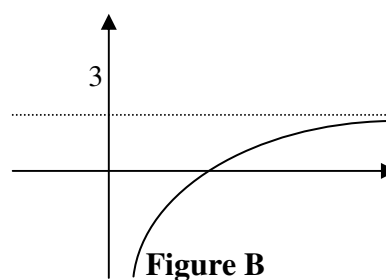


**Figure 4.4 The limit is reached**

However, there are some inconsistencies in students' assertions (Mutemba, 2001). Mutemba presented Figures A and B to Grade 12 students.



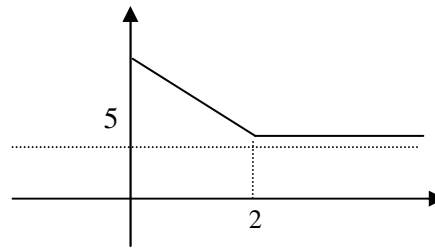
**Figure A**



**Figure B**

Some of them claimed that in Figure A the limit was 3, but not in figure B, because the function does not assume this value.

On the other hand, when she asked them to sketch the graph of a function given the condition  $\lim_{x \rightarrow 2} f(x) = 5$ , some of them sketched the following graph:



**Figure C**

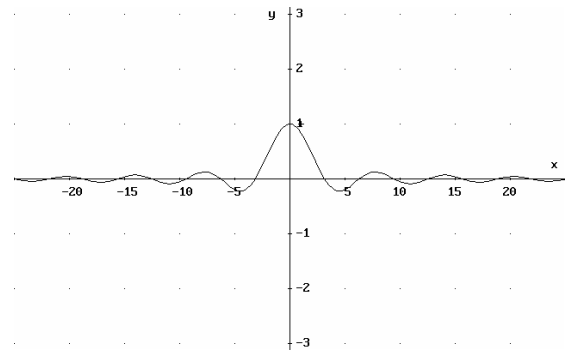
These students seem to have the idea that the limit cannot be reached.

Another strong idea frequently observed in first year university students is that, as happens with the vertical asymptote, the horizontal asymptote cannot be crossed.

For example, in the case of the function

$$f(x) = \frac{1}{x} \sin x, \text{ that goes to } 0 \text{ when } x \text{ goes}$$

to infinity, crossing the asymptote  $y = 0$  an infinite number of times, they would not consider the  $x$ -axis as an asymptote and it would be difficult to them to read the limit from the graph.



**Figure 4.5**  $f(x) = \frac{1}{x} \sin x$

Mutemba asked students to read the limit when  $x$  increases in a similar case. Most of them tried to make algebraic transformations or to get a formula of pairs of values to indicate the limit, when they were given a graph. She concludes that they held a concept image of a limit as a procedural method (2001: 60).

In addition to these misconceptions related to the graphic representation of functions, teachers should be able to analyse more complicated situations, even if they do not appear at secondary school, as for example the function defined as follows:

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$$

In this case it is not possible to represent the function on a graph, but the limit

when  $x$  goes to infinity does exist because  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ .

Some researchers pointed out epistemological obstacles related to limits of functions (Cornu, 1982, 1983, 1984, and 1991; Sierpinska, 1985). I will elaborate here Cornu's categories. Sierpinska reinforces these findings.

Cornu argues that, although students do not need to go through the historical way of the concept formation, there are clearly some similarities between what happens with learners now and what happened historically. Through a didactical sequence constructed in order to find out the obstacles faced by students, he identified several epistemological obstacles. I will focus on some of them which seem more important in the Mozambican context.

- *The failure to link geometry with numbers*

The Greeks used limits implicitly for solving geometrical problems, for example using the exhaustion method which seems very close to the limit concept. However

There is no transfer from geometrical figures to a purely numerical interpretation, so the unifying concept of limit of numbers is absent. The geometrical interpretation, and its success in solving pertinent problems, is therefore seen to cause an obstacle which prevents the passage to the notion of a numerical limit. (Cornu, 1991: 160)

In Mozambique, according to the syllabus, the first intuitive contact of students with the limit concept, in Grade 7, is in a geometrical context, when studying the area of a circle. I do not think that this fact can be an obstacle for studying this concept in a numerical setting. On the contrary, linking the two points of view could help the student construct a richer concept of limits.

- *The notions of infinitely large and infinitely small*

Throughout the history of the notion of limit we meet the supposition of the existence on infinitesimally small quantities. It is possible to have quantities which are so small as to be almost zero, and yet not having 'assignable' size? What happens at the instant when one of these quantities becomes zero? (Cornu, 1991: 160)

Many students conceive  $\varepsilon$  as a number smaller than the "real" numbers but nevertheless not zero, and infinity as a number greater than the "real" numbers, but

nevertheless not exactly infinite (Cornu, 1983: 255). In fact, in Mozambique, many students operate with infinity as with a real number. For example, they would write:

$\lim_{x \rightarrow \infty} (x^2 + x + 1) = (\infty)^2 + \infty + 1$ . This leads to situations such as  $+\infty - \infty = 0$  or

$\frac{\infty}{\infty} = 1$ . What conception of infinitely large and infinitely small do teachers hold?

How can they help their students understand these concepts?

- *The metaphysical aspect of the notion of limit.*

The infinity and the limit concepts seem to be more related to metaphysics than to mathematics; infinity is mysterious. For students who are used to calculating algebraically, it is difficult to understand what the limit of a function is. “How can we be sure that a number exists, if we cannot calculate this number?” (Cornu, 1984: 255). Are teachers aware of this difficulty?

- *Is the limit attained or not?*

According to Cornu,

this is a debate which has lasted throughout the history of the concept. For example, Robins (1697-1751) estimated that the limit can never be attained, just as regular polygons inscribed in a circle can never be equal to the circle. (1991: 161)

For him “this debate is still alive in our students” (1991: 162). We have already seen that this problem arose through Mutemba’s study.

- *The problem of the transition from finite to infinity*

Cornu contends that the students held

a static view, which hinders a more dynamic view, and in which what happens ‘within the finite’ allows to anticipate what happens ‘towards infinity’, and therefore to speak about limit.

[une vision statique, qui fait obstacle à une vision plus dynamique, dans laquelle ce qui se passe ‘dans le fini’ permet de prévoir ce qui se passe ‘à l’infini’, et donc de parler de limite] (1983: 256).

Cornu (1983) argues that these obstacles are not organised in series but are connected in a very complex way.



The obstacles faced by students when learning the limit concept, even if they have common points, may differ according to the culture and language in which the study is done. For these reasons it is important to analyse students' difficulties in the context of Mozambican secondary schools. These difficulties have strong links with the essential features of the concept and the different representations in which the limit is considered.

To conclude, we can say that students face several difficulties in working with or in giving meaning to the limit concept. A previous study (Huillet & Mutemba, 2000) showed that Mozambican teachers are generally aware of some of these difficulties, particularly the difficulties in understanding the formal definition and the mistakes that students usually do when calculating indeterminate forms, but that none of the teachers indicated other difficulties, particularly those linked with the understanding of the concept. Teachers probably face the same problems in giving meaning to the limit concept, which makes it difficult for them to mediate students' learning.

## **4.7 Conclusion**

In this chapter I analysed the mathematical knowledge on limits of functions needed by Mozambican teachers to perform the second step of the didactical transposition when teaching this concept (Mathematics for teaching limits). I did it considering that they need to be aware of the first step of the didactical transposition already carried out by the institution, including the justification for teaching this concept at that level, as well as the tasks that they have to perform when planning their lessons. This chapter also illuminates what the relation to the limit concept of the institution to be created ought to be in order to make teachers' personal relation to the concept evolve.

In the next chapter I present this new institution. I argue that teachers' activities within this institution are expected to change their personal relation to the limit concept, and I describe how this institution has been set up.

# CHAPTER 5

## THE NEW INSTITUTION

## 5 The New Institution

In Chapter 2, I presented two Mozambican didactic institutions' relation to the limit concept: Secondary schools and the Pedagogical University. I argued that Mozambican teachers' relation to this concept had been shaped by the institutions where they met this concept. As a consequence they would probably teach it according to these institutional routines and would have difficulties in questioning these routines. My position is that changing the usual teaching practices is only possible if teachers' personal relation to limits evolves.

In Chapter 3, I argued that teachers' mathematical and pedagogical knowledge is intertwined and I developed a framework for *mathematics for teaching* (MfT), based on an analysis of teachers' activities when building a new mathematical organisation.

In Chapter 4, I applied this framework to the analysis of MfT limits of functions in Mozambican schools in a more elaborated way, and explained how research in mathematics education could inform this teaching. Obviously I do not claim that a change in teachers' personal relation to limits will automatically result in a change of their way of teaching this concept in schools. Other institutional, and even personal, constraints could lead them not to change their usual teaching ways, although being aware of its limitations. My argument is that their actual personal relation to limits, moulded by the Mozambican institutions' relation to this concept, does not allow them to challenge the didactical transposition currently carried out in secondary schools. The evolution of their knowledge is a necessary but not sufficient condition for any change in teaching limits in Mozambican secondary schools.

From an anthropological point of view, teachers' knowledge can only evolve if they are in contact with limits through an institution whose relation to this concept is different from the usual one. I therefore needed to set up a new institution whose relation to limits of functions allowed for the development of *mathematics for teaching* limits as described in Chapter 4.

Through what kind of institution could teachers develop their knowledge on the aspects of the limit concept presented in the previous chapter?

In line with the conception of *knowledge-of-practice* (see page 61), I regard teachers as co-constructors of their own knowledge, and not as mere consumers of knowledge elaborated by other people, such as academics. I therefore needed to place some teachers in a position to produce some knowledge about limits of functions. Several experiences in teachers' development (see following sections) show that teachers improve their knowledge and practice through research and through interaction with colleagues. During the specific examples described in section 5.1, which have their roots in the action research movement, teachers usually researched some aspect of their classroom practice or of their students' difficulties. My own study does not focus on teachers' practice, but on their personal relation to the limit concept. Nevertheless, I suggest that this personal relation can also evolve through research, and through interaction between teachers sharing their results and difficulties during the research process. I thus selected to set up the new institution as a "research group" looking at limits of functions as a mathematical object to be taught in Mozambican secondary schools. This means that each teacher would research a different aspect of this concept, analyse the possible uses of this new knowledge in teaching at secondary school, and share his/her findings with his/her colleagues. The topics to be investigated should be chosen in order to develop the aspects of MfT presented in the previous chapter.

These are the issues that I develop in this chapter, which is structured as follows:

- 5.1. Mathematics teachers as researchers
- 5.2. Setting up a new institution
- 5.3. The research topics
- 5.4. Conclusion

## 5.1 Mathematics teachers as researchers

In this section I briefly present the teachers-as-researchers movement and some debates which arose from this movement. I then look at how it has developed in mathematics education and its implication for my own research.

### 5.1.1 The teachers-as-researchers movement

According to Elliott (1991), the teachers-as-researchers movement emerged in England during the 1960s, in the context of curriculum reform. Initially it focused on the teaching of humanities subjects, teachers working together in cross-subject teams. The research was not systematic, but occurred as a response to particular questions and issues as they arose. It aimed to improve practice rather than to produce knowledge.

This movement extended in the 1980s into what is usually called *the teacher research movement*. In a paper called “The Teacher Research Movement: A Decade Later”, Cochran-Smith & Lytle (1999) review papers and books published in the United States and in England in the 1980s disseminating some experiences of teacher research. They argue that

the visions of educational research embedded in these writings shared a grounding in critical and democratic social theory and in explicit rejection of the authority of professional experts who produced and accumulated knowledge in “scientific” research settings for use of others in practical settings (1999: 16).

This seems to be the main feature of the teacher research movement. Within this movement, teachers are no more considered as mere consumers of knowledge produced by experts, but as producers and mediators of knowledge, even if it is local knowledge, to be used in a specific classroom or in a specific school. In fact, in most of their research, teachers focussed on their own classroom practice. Most of these experiences seem to rely on a dichotomy between mathematical content and pedagogical practices, and could be considered as aiming to develop *knowledge-in-practice*, as described by Cochran-Smith & Lytle (see page 60) and not mathematical knowledge for teaching.

Many debates arose about the teacher research movement. They focus mainly on the problem of whether it can be considered research or not, considering that some of the projects carried out by teachers do not fill the requisites of formal research, such as systematic collection and analysis of data, and dissemination of the research's results (Richardson, 1994; Cochran-Smith & Lytle, 1990 and 1999; Breen, 2003).

Richardson argues that teacher research is a “confusing concept” (1994: 6), as there are several motivations captured in this notion. She distinguishes two forms of teacher research on practice: practical inquiry and formal research.

According to Richardson, several approaches can be qualified as practical enquiry, such as *teacher as reflective practitioner* and *action research*. This kind of research does not aim to produce general results concerning educational practice, but to suggest new ways of looking at the context and possibilities for changes in practice. It produces local knowledge for purposes of improvement in one's everyday life and is not generally disseminated. Formal research means to contribute to a larger community's knowledge.

Other authors only use the term *action research* to design a more systematic, self critical enquiry that has been made public (Adler, 1992; Brown, 1997). According to Crawford & Adler

The term *action research* is now widely used to describe investigations and inquiry undertaken with an intent to change professional practice or social institutions through the active and transformative participation of those working within a particular setting in the research process. A major aim of most action research projects is the generation of knowledge among people in organisational or institutional settings that is actionable - can be used as a basis for conscious action. (1996: 1187)

This more formal research has usually been conducted by teams involving teachers and educational researchers.

I will not go further into these debates about teacher research. I acknowledge that there are basically two kinds of research: formal research which aims to contribute to a large mathematics education community's knowledge, and less formal research usually done by teachers and which aims to produce local knowledge and improve

teachers' practice. To make it simpler, I will use the term "research" for both formal research and inquiry about practice without distinction.

### **5.1.2 Teacher research in mathematics education**

In Mathematics Education, the teacher as researcher movement has now become an important part of many teacher education programs all around the world. It also has been the topic of debate within the mathematics educators' community and of several papers presenting results of these programs or discussing some aspects of teacher research. Most of these publications focus on teachers' practices.

In 1988, the International Group for the Psychology of Mathematics Education (PME) started a working group called "Teachers as researchers". This group met annually for nine years and published a book based on contributions from its members (Zack, Mousley & Breen, 1997). This book presents different experiments, made in several countries and using several methods, with the aim of developing practice through teachers' inquiry.

Adler (1992) reports the case study of a middle-class mathematics teacher researching his interactions with learners and their interaction with each other, during his post graduate study. Through this research, he realised that he dominated classroom interaction and that his mediation was gendered.

Breen reports the experience of the Mathematics Education Project (MEP) which attempts to get primary teachers "to move from being a passive passer-on of knowledge" (1997: 151). They were encouraged to write about their process of change and to share extracts from each other's journal. He brings up the problem of the equality of role in the partnership between the university researcher and the teachers involved in the project.

D'Ambrosio (1998) relates two experiences of learning through teacher research.

In the first, pre-service secondary mathematics teachers formed a research team which investigated children's understanding of fractions. D'Ambrosio reports a growth in students' reflective thinking that teaching experience alone would not generate.

In the second experience, teachers were encouraged to identify a research question related to their classroom practice through personal journals. A certain pattern emerged in teachers' choice: several teachers chose to look at how to manage their classroom better; others chose to study a student or a small group of students and their learning. Every week these teachers presented their findings to their small working group. D'Ambrosio concludes that "the teachers who engaged in teacher research found themselves questioning their practice and wondering and planning what they might do differently" (1998: 155).

Hatch & Shiu (1998) reports the case study of a primary school teacher researching her own practice through the analysis of class transcript and a reflective journal as part of an in-service course. They argue that she contributed not only to developing knowledge of her own practice but potentially to the accumulated knowledge of the research community.

Halai (1999) reports on action research conducted by mathematics teachers in Karachi, and involving university researchers as facilitators. They also used participant observation, field notes, and reflective journals. She claims that this action research project promoted learning and professional growth not only of the teachers but also of the university researchers.

Edwards & Hensien (1999) describes action research collaboration between a middle school mathematics teacher and a mathematics teacher educator, involving observation and discussion of lessons and exchanging roles in the classroom. The analysis of the teacher's narrative of this collaboration as well as the teacher's regular reflections on her beliefs and practices were important to her process of change.

Jaworski (1998) describes the MTE (Mathematics Teacher Enquiry) project, which involved six secondary mathematics teachers undertaking their own research independently of an academic programme. These teachers were invited to identify some question they were interested in investigating. Jaworski points out that, during this research, the teachers focused their attention on pedagogical issues, rather than on mathematical issues.



Decisions about what mathematics should be done, what classroom tasks would be appropriate, and what outcomes would be desired, were a normal part of the teaching process, hard to extract as problematically related to the research issues. (1998: 25)

She asks the question "How might mathematics issues become more overt in the research project?" (1998: 29).

This is one of the questions that I had to lead with in my own research.

In fact, in most of the papers presented above, the focus was on teachers' classroom practices, independently of the knowledge to be taught. In one of them (D'Ambrosio, 1998), teachers investigated a specific content, fractions, focusing on students' difficulties. In all these projects, it seems that the mathematical content to be taught is taken for granted, and that teachers are not supposed to challenge it. They are only supposed to try to improve their teaching practices.

I found a few articles which mention some change, or some possible change, in teachers' knowledge of mathematics.

Mousley (1992) reports the results of one year course in an off-campus mode called MATHEMATICS CURRICULA. Course participants chose one or several areas to change their practice, and used cycles of action research until "they feel that the innovation sits comfortably within their routine patterns of teaching" (1992: 138). They were required to work with colleagues. A representative sample of sixty teachers was then contacted by letter, telephone or by personal interview about the impact of the course. It was found

not only some ongoing restructuring of pedagogy, in terms of content, organisation and classroom interaction, but also growth of understanding about (1) the nature of mathematics, (2) the processes of teaching and learning of maths, (3) the power of institutional contexts of teaching and learning, and (4) the processes of pedagogical change. (Mousley, 1992: 138)

Although the aim of this project was to improve practice, it also shows that, through their research, teachers' knowledge of mathematics evolved, and that they became aware of the weight of institutional constraints.

The notion of mathematics as a stable body of knowledge and skills to be transmitted and practiced became problematic. Questioning traditional classroom practices provided an incentive for teachers to confront given curriculum content. (1992: 139)

Mousley concludes that

participatory, experience-based research has the power to emancipate some teachers from taken-for-granted classroom routines which constrain and control mathematical learning. The dialectical interaction of reflection combined with social interaction allowed innovation in the nature of teaching and learning mathematics as well as in curriculum content. (1992: 143)

This experience shows that through research and interaction teachers can be led to challenge an institutional relation to mathematics.

In the first edition of the *International Handbook of Mathematics Education*, Crawford & Adler (1996) suggest:

It seems possible if teachers and student-teachers act in generative, research-like ways, they may learn about the teaching/learning process, and about mathematics, in ways that empower them to better meet the needs of their students. (1996: 1187)

In this quote, these authors seem to avoid the distinction between practical inquiry and more formal research, using the term “research-like ways”. The focus is on teachers’ personal learning through research, not only about their practice, but also about mathematics. They argue that, as the quality of teachers’ mathematical knowledge is strongly influenced by their own experience as students, they need to unlearn the old conceptions of mathematics deriving from their schooling experience. The experiences of “teachers’ voices” in South Africa and of a program of action-research with student teachers in Australia lead Crawford & Adler to conclude that research helps teachers challenge their practice and their conception of mathematics. Student teachers doing action research “learn a great deal about mathematics as they work with their students to define and refine mathematical ideas and use them actively as a means to inquiry” (1996: 1200).

Another project research reporting changes in teachers’ knowledge of mathematics is the PLESME project, where mathematical knowledge and

mathematics pedagogical knowledge were intertwined. It has been described in section 3.1.2 (see pages 61-62).

Speaking about this project, Graven (2005) explains:

I spent much time resisting teachers' expectations that I knew what the ideal 'new curriculum' was and could and would explain it to them. This is not to say that I did not have my *own preferences or principles of selection* that influenced the nature of the workshops, the methodologies that I drew up for workshops, the comments I made on teachers' lessons and the nature of PLESME activities. [...] I experienced a tension between making explicit to teachers the principles (values) I was drawing on and my preferences for teaching, while at the same time holding back judgment and notions of 'best practice'. (2005:224)

This was also the challenge for me. As I described in Chapter 4, I had strong ideas about the way limits of functions could be taught in schools. However, I did not want to provide the teachers with my own ideas, but to help them reflect on the limit concept as a mathematical object, on the actual practice of teaching this concept in secondary schools, even if it was not their own practice, and to develop new ideas about this practice.

As a result of the PLESME programme, Graven asserts that

teachers challenged the 'all-knowing' construction of 'a professional teacher'. This new construction supported teachers in strengthening their identities as mathematics teachers despite the limitations of their pre-service studies (2005:225).

And Graven concludes: "The most important outcome of INSET should be enabling teachers to adopt identities as lifelong learners that endure far beyond the scope and life span of the INSET." (2005:225)

I fully agree with this conclusion, which resonates with my own project. The new institution to be set up, in addition to its role in the evolution of teachers' knowledge on limits of functions and its teaching in Mozambican secondary schools, should enable them to think of themselves as life-long learners. One of the desirable consequences of their participation in the project would be the enabling of teachers' reflections on their teaching of other topics and the

challenging of the didactical transposition of these topics performed by the institution where they teach.

### **5.1.3 Overview**

This non exhaustive review of papers about the teachers-as-researchers movement shows the diversity of experiments done in this domain, in terms of research topics and methodology. However, some common trends can be identified.

In the first place, they seem to share a common conception of teacher as a producer of knowledge and not as a mere consumer of knowledge produced by other individuals, particularly academics. This is in line with the knowledge-of-practice conception presented in Chapter 3. It is also the conception adopted in this study. Teachers' personal relation to limits of functions is expected to be strongly rooted in the traditions and practices of Mozambican didactic institutions. In order to challenge this view, and to effect a change in institutionalized routines, they need to engage in deep reflection on this topic, and I claim that research would be a good method to do that. Several mathematics educators point out the fact that teachers do not usually use the findings of research. According to Cochran-Smith & Lytle, "teachers often find it irrelevant and counterintuitive" (1990: 4). They would probably consider findings coming from their own research, and grounded on their own experience and practice, more relevant and useful for their own practice.

Furthermore, in most of these research projects, teachers worked together in groups, the research team being composed of either pre-service or in-service teachers. Interaction between teachers, or between teachers and mathematics educators, allowed them to deepen the analysis of their practices and difficulties. In my study, teachers were expected to challenge the institutionalised tradition of teaching limits in schools. This would be difficult as an individual project, or if restricted to interaction with me as a supervisor. Sharing their own difficulties when learning limits, or when teaching limits in the case of experienced teachers, discussing other aspects of the limit concept and other possibilities of teaching it in schools, would open far greater possibility for teachers to negotiate new

meanings of this concept and to look at the current teaching in a more critical way.

Finally, in all of the projects discussed in this chapter, teachers chose to investigate some pedagogical issue or some problem of student learning. Pre-service teachers worked in a research team to investigate some classroom problem. In the case of in-service teachers, the teachers were usually engaged in research about their classroom activities. It seems that when asked to choose a research topic, teachers question their own teaching, or their students' performance and difficulties, but take for granted the content usually taught within the institution. Cochran-Smith & Lytle also observe that teachers' questions "often emerge from discrepancies between what is intended and what occurs" (1990: 5). I did not find any example of teachers' research challenging "what is intended".

My argument is that changing the way of teaching the limit concept in Mozambican schools is not as much a question of teaching methods as a problem of content. The Mozambican secondary schools' institutional relation to this concept is mainly algebraic, and could be expanded, for example by using different representations and applications, as I showed in Chapter 4. This could only be possible if the teachers acknowledge that their conception of the limit concept is restricted, and that they need to expand their knowledge to other aspects of this concept, and analyse other possibilities of using these new facets of limits in their teaching.

In summary, the results of several experiences of teachers' research show that teachers learn a lot about their own practices and about students' difficulties through research and through interaction within a research group. I argue that their mathematical knowledge would probably evolve by researching institutional practices instead of personal practices, and by sharing their findings within a group. Improving their mathematical knowledge, they will be able to challenge the didactical transposition of the limit concept usually done in Mozambican secondary schools. The design of the new institution was grounded in this assumption.

## 5.2 Setting up a new institution

As reported in the last section, teachers appear to improve their practice and their knowledge of students' difficulties through research and through interaction within a research group, and I argued that they would also improve their mathematical knowledge using the same method. Obviously the kind of research undertaken by the teachers should be different in nature from the topics usually chosen by teachers for their research. It should not be formal educational research or inquiry about practice as in action research, but more mathematical research, although oriented towards teaching.

All these considerations led me to the idea of setting up a new institution where several teachers research different aspects of the limit concept, and share and discuss their findings during periodic seminars. To bring this idea to life, I faced two main problems.

The first problem was to find teachers interested in joining such a research group. Why would a Mozambican secondary teacher, who is already overloaded with teaching in several institutions, accept participation in a study group about the limit concept without having a benefit from this work?

The second was to define research topics. These topics should be chosen in order to develop the aspects of *mathematics for teaching* the limit concept, as described in Chapter 4. Therefore I had to develop these topics myself and suggest them to the teachers. I will go into the details of these topics in the next section.

In this section, I explain how I dealt with the first of these problems: how to set up a group of teachers willing to participate in a research group about limits of functions.

Speaking about classroom-based research, Setati (2000) draws attention to the fact that researchers are often viewed as using teachers and students for their own benefit. She argues that teachers who agree to participate in a project also come in with agendas. This also applied to my project and I thought that I could overcome this difficulty by working with student teachers who needed to do research for their degree.

When I began my research, secondary school teachers were trained in bivalent five-year courses at the Pedagogical University (PU)<sup>12</sup>. Mathematics teachers were trained to teach both mathematics and physics. During their last year of training, they were required to write a dissertation on a research project in mathematics or physics education. Through conversations with colleagues from the Pedagogical University, I knew that some student-teachers did not conclude their courses because they did not have a supervisor to help them with their research. PU lecturers were already overloaded with classes and other activities, and did not have time to assist all the students who needed to be supervised in their research. In the meantime, most of these students were already teaching in secondary schools, although usually not at pre-university level, where the limit concept is taught. Including some of these students in my research group could help them conclude their training course. In that way they would have their own agenda for joining the group. Furthermore, these students sometimes faced difficulties in accessing a computer for their work, and I could place a computer at their disposal in my room at Eduardo Mondlane University (EMU). I could also help them with papers about limits that I collected over the years for my own work.

During the same period, the Faculty of Education at EMU was starting a Masters Degree in mathematics and science education. Most of the students were experienced teachers, some of them mathematics teachers. They also had to write a dissertation, in that case corresponding to a more formal research project in mathematics education. The dean of the Faculty agreed that I could ask for volunteers to participate in my research group.

Given these possibilities, I decided to form a group of six teachers, some of them from the last year PU course and others from the Masters Degree at EMU Faculty of Education. I expected the Masters Degree students to have some experience in teaching limits of functions at secondary school. This experience could enrich the discussions within the group. All of them would have to write a dissertation on one specific aspect of the limit concept, under my supervision. In addition to this individual work, periodic seminars would be held, where these teachers could

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<sup>12</sup> The courses changed at the Pedagogical University in 2006. They are now one subject three-year courses.

share their ideas and findings, or discuss some specific feature of the limit concept. In that way they would access other aspects of limits, and develop their knowledge and understanding of this concept through interaction with colleagues.

Using Chevallard's terminology, the new institution that I intended to set up can be considered as a didactic institution, as its intention was that its subjects develop a more elaborated relation to the limit concept. However, this institution would have several components. First of all, inside this institution, each subject would be doing research, which is an individual task. To perform this task, they would have my support as their supervisor, and the support of another supervisor in the case of the Masters Degree students. This would take the form of individual supervision sessions of each participant with his/her supervisor(s). Furthermore, they would attend periodic seminars, where they could get support from the other colleagues of the research group, learn from colleagues' research work, give their opinion about these colleagues' work and debate specific aspects of limits of functions. In addition to these activities, I decided to hold individual interviews with each teacher involved in the group, in order to collect more information about the evolution of their personal relation to limits.

Considering the components of MfT limits in schools (see Chapter 4), I needed to determine the research topics for the teachers involved in my research. This would ensure that all these aspects were included. Furthermore the research topics ought to be suitable research topics for an Honours or a Masters Degree dissertation.

### **5.3 The research topics**

As I wanted to work with six teachers, I defined six research topics. These topics do not correspond exactly to the aspects of MfT limits presented in Chapter 4 for two main reasons.

In the first place, and as I explained in Chapter 1, my framework evolved during the research process. On the one hand, it was informed by new presentations and applications of Chevallard's theories (Chevallard, 2002a and 2002b ; Barbé et al., 2005) and from the results of discussions held within the French mathematics educators community about Douady's concepts of setting and change of setting (Duval, 1999). On the other hand, the analysis of my own data also led me to a



new analysis of this framework. For this reason, the topics defined for teachers' research at the beginning of the process are in line with my first analysis of Even's categories of SMK for limits of functions, and not with the categories that emerged from further analysis. Nevertheless, the categories that I developed later, based on Even's categories analysed through the lens of the didactical transposition, would have led me to similar topics for teachers' research. I will present these topics as they were defined at that time, indicating when necessary what could have changed if I had used the categories developed later.

Secondly, the facets developed by Even refer to components that allow us to analyse the knowledge teachers have (or should have) about a concept. There are also the aspects of teachers' personal relation to limits of functions whose evolution I wanted to study through their participation in the research group. Some of them did not seem suitable for a dissertation, such as "essential features", while other topics, such as the history of the limit concept, appeared to be very useful to access several aspects of the concept.

In accordance to these constraints, I selected the following topics for the teachers' research:

- The history of the limit concept and its implications for teaching;
- The use of different settings and models to teach the limit concept;
- Alternative ways of introducing the limit concept in schools;
- Applications of limits of functions in mathematics and in other sciences;
- Basic repertoire for teaching limits of functions in schools;
- Students' conceptions and difficulties when learning the limit concept.

I will now present each topic as it was defined at the beginning of the research process and its relations to the different aspects of MfT limits.

### 5.3.1 The history of the limit concept and its implications for teaching

The aim of this topic was to answer the following questions:

- How does the concept “limits of functions” emerge in the history of mathematics?
- How did it evolve into the definition currently used by mathematicians?
- What can we learn from the history of this concept that could be helpful for its teaching and learning at secondary school level?

As we could see in the analysis of MfT limits of functions (see Chapter 4), the epistemological study of the limit concept allows us to access many aspects of this concept, which can help the development of teachers' personal relation to limits.

In the first place, the history of the limit concept shows the different features it has assumed over time and that compose its main features today: the dynamic point of view, the static point of view and the operational point of view.

Moreover, the historical study of limits gives us information about the different settings in which the concept has been studied over time: first geometrical and numerical, then functional and algebraic, and later numerical again and topological.

The history of the limit concept also informs us about the problems, and how their resolution led to the development of this concept: geometrical problems (length of a curve, areas and volumes), cinematic problems (determination of instantaneous velocity and acceleration), functional problems (construction of the tangent line), maximum and minimum problems, etc. Some of these problems can be used to introduce limits in schools, but also to construct tasks that will enrich the teachers' basic repertoire (practical block).

The difficulties faced by mathematicians over time can also help understand the difficulties that students face when learning this concept.

Furthermore, the study of history helps us understand that mathematics is not a static body of knowledge but a science which develops through conjectures, proofs and refutations. It will contribute to deepen teachers' knowledge about

mathematics: the role of definitions, the role of proofs, and the connectedness of knowledge.

In conclusion, the study of the history of the limit concept was expected to develop all the aspects of MfT limits.

### **5.3.2 The use of different settings and models to teach the limit concept**

In terms of the framework for MfT developed in Chapter 3, this should be called “The use of different settings and *registers* to teach the limit concept”. I used the term *model* at the beginning of this study, and this is the word that has been used by the teachers during the research process.

The questions to be addressed with this topic were:

- In which settings can the limit concept be studied and which different models (registers) can be used in each setting?
- How can we use these different settings and registers in secondary schools, particularly the changes of settings and registers, in order to reach a deeper and richer understanding of the concept?

In addition to the knowledge about settings and registers, this study ought to contribute to develop the teachers' knowledge about:

- Essential features, that emerge from the different settings;
- Alternative ways to introduce the concept, that can come from different settings;
- The strength of the concept, by showing different aspects of limits;
- The basic repertoire that can be used in secondary schools (practical block), by considering tasks in different settings and tasks to shift from one representation to another;
- Students' conceptions and difficulties, because working in different settings helps students broaden their conception, and changes of register help teachers understand students' difficulties;

- Mathematics in general, in particular the connectedness of mathematical knowledge, the role of definitions, and the use of symbols.

As with the history of limits, this topic was expected to develop all aspects of MfT this concept.

### **5.3.3 Alternative ways of introducing the limit concept in schools**

In Chevallard's terms, this could be called "Organising students' first encounter with the limit concept".

The aim of this topic was to answer the following questions:

- Which are the several ways of approaching limits of functions at secondary school?
- Which could be the consequences of these different approaches for students' concept image (Tall & Vinner, 1981)?

This topic was directly related to the organisation of students' first encounter with limits. It should also help to develop the following other aspects of MfT limits:

- Essential features, because some of the various ways of introducing limits correspond to different points of view about limits;
- Different settings and registers, because the different ways of introducing limits belong to different settings and may use different registers;
- The strength of the concept that emerges from these different approaches;
- Basic repertoire, because some of the ways of introducing limits of functions can also be used as tasks for students;
- Knowledge about mathematics, in particular its connectedness.

As with the previous topics, this topic was expected to develop all aspects of MfT this concept.

### **5.3.4 Applications of limits of functions in mathematics and in other sciences**

This topic aimed to answer the following questions:

- What are the applications of the concept “limits of functions” at secondary school level, in mathematics and in other sciences?
- What are the applications in mathematics at university level?
- How can we use these applications in the Mozambican secondary school context?

The study of different applications of the limit concept, in mathematics or in other sciences, was expected to develop teachers’ knowledge about:

- Different settings and registers, because the applications of this concept belong to different settings, and even to different areas of knowledge such as biology, economics or physics;
- Alternative ways of introducing limits, because some of the applications can also be used to introduce the concept;
- The strength of the concept, that emerges from a great variety of applications;
- Basic repertoire, because some of the applications can be used as tasks for secondary school students;
- Students’ conceptions because different applications lead to different conceptions;
- Mathematics, because knowing several applications of the limit concept helps to develop the idea that mathematical knowledge is useful and connected.

This topic was expected to develop most aspects of MfT this concept. The only aspect which does not have a direct link with this topic is “essential features”.

### **5.3.5 Basic repertoire for teaching limits of functions in schools**

This topic relates to the practical block of the MO about limits. It aimed to answer the following questions:

- What should be the basic repertoire for the teaching and learning of limits of functions in schools?
- What influence would each task in this basic repertoire have on the learning of this concept?

Besides the development of teachers' knowledge of the basic repertoire, this topic ought also to contribute to develop their knowledge about:

- Different features that can emerge from a great variety of tasks;
- Different settings and registers, because the basic repertoire must be grounded in different settings and in tasks for shifting from one register to another;
- Alternative ways of introducing limits, because some tasks can also be used to introduce the concept;
- The strength of the concept that was expected to emerge from the variety of tasks considered in the repertoire;
- Students' difficulties, because when constructing tasks for secondary school level we must think about the difficulties students will face when solving them;
- Mathematics, because to reflect on a basic repertoire we need to reflect on the role of definitions, the roles of proof and the connectedness of knowledge.

As with the first three topics, this topic was expected to develop all aspects of MfT limits.

### **5.3.6 Students' conceptions and difficulties when learning the limit concept**

This topic aimed to answer the following questions:

- What are the main difficulties that students face when learning the concept "limits of functions" at secondary school and at university in Mozambique (for example at Eduardo Mondlane University or at the Pedagogical University)?
- What conception of the limit concept do they hold?

This topic would also lead to reflect on the following aspects:

- Essential features, because students' conceptions are linked to different features of limits;
- Different settings and registers, because students' difficulties may depend on the different representations they are working with;
- Alternative ways of introducing the concept, because different forms of introduction may lead to different conceptions and difficulties;
- The basic repertoire, because the conceptions and difficulties may depend on the tasks that students solve in schools;

This topic was expected to develop most aspects of MfT this concept. However it does not have a direct link with the “strength of the concept” and “knowledge about mathematics”.

### **5.3.7 Overview**

As I have already stated, the six topics for the teachers' research have strong links across topics and with the aspects of MfT limits of functions presented in my framework. It was expected that these links would appear during the periodical meetings of the whole group, and that the conversations within the group would enable the teachers to access the limit concept as discussed and researched by others. Table 5.1 (pages 153-154) presents the six topics proposed for the teachers' research, related to the aspects of MfT limits that they were expected to develop through their personal work and also through the discussions within the group.

## **5.4 Conclusion**

In this chapter I presented some experiences of teachers' research. I showed that these experiences shared a common conception of teachers as producers of knowledge and not mere consumers of already produced knowledge. However, the knowledge produced by this kind of research is basically pedagogical knowledge. Teachers usually research their own practices or students' difficulties, and in most of the cases this does not lead them to challenge the institutional relation to mathematical knowledge, which is taken for granted.

My analysis of Mozambican secondary school's relation to limits of functions (Chapter 2) led me to consider that it could be expanded (Chapter 4). This could only be possible if teachers challenged this institutional relation, extending their own personal relation to this concept. For this purpose, a new institution needed to be created, where teachers could improve mathematical knowledge for teaching limits through research on several aspects of limits and sharing their findings within the group.

In the next chapter I describe in more detail the methodology used to set up this institution, as well as to collect and analyse data coming from the teachers' individual research, the seminars and the interviews.



**Table 5-1 Relation between research topics and aspects of *Mathematics for Teaching* limits**

<div>Research topics</div> <div>MFT limits of functions</div>	The history of the limit concept and its implications for teaching	The use of different settings and models to teach the limit concept	Alternative ways of introducing the limit concept in schools	Applications of the limit concept in mathematics and in other sciences	Basic repertoire for teaching limits of functions in schools	Students' conceptions and difficulties when learning the limit concept
<b>Essential features</b>	Different features appeared over time that compose the main features of limits today	From different settings emerge different features	Different ways of introducing limits correspond to different features	<i>No direct link with essential features</i>	Different features can emerge from a great variety of tasks	The conceptions of students are linked to different features of limits
<b>Different settings and registers</b>	The limit concept has been studied in different settings over time	Direct link	Different ways of introducing the limit concept belong to different settings	The applications belong to different settings	A basic repertoire is grounded in different settings and in tasks designed to shift from one register to another	Students face different difficulties in different settings
<b>Alternative ways of introducing the limit concept</b>	Different problems originated the development of the limit concept and can be used to introduce it	Different settings can be used to introduce limits	Direct link	Some of the applications can be used to introduce the concept	Some tasks can also be used to introduce the concept	Different forms of introduction lead to different conceptions and difficulties
<b>Strength of the concept</b>	The struggle of mathematicians when studying this concept and its applications over time in maths and in other sciences shows its strength	The fact that limits can be studied in many settings reinforce its strength	The strength of the concept emerges from different forms of introduction	The great variety of applications shows the strength of the concept	The strength of the concept emerges from the great variety of tasks	<i>No direct link with the strength of the concept</i>

<b>Basic repertoire</b>	Different settings and different problems whose resolution originated its development can be used to construct tasks	Tasks can be elaborated either in different settings or in order to shift from one representation to another	Some forms of introduction can also be used to construct tasks	Some of the applications can be used as tasks for secondary school students	Direct link	Students' conceptions and difficulties depend on the tasks they solve in schools
<b>Students conceptions and difficulties</b>	The history of limits can help understanding some students' difficulties	Changes of representation help understand students conceptions	Different ways of introducing the topic can lead to different conceptions of limits and must take into account students' difficulties	Different applications can lead to different conceptions	To construct tasks we must reflect on the difficulties students would face when solving them	Direct link
<b>Knowledge about mathematics</b>	The study of the history helps to understand the mathematical work, the role of definitions and proof, the use of symbols and the "connectedness" of mathematical knowledge	Different settings and registers are an important part of the knowledge about mathematics	Knowing that there are several ways to introduce a concept is part of the knowledge a teacher should have about mathematics	Knowing several applications of the limit concept helps to develop the idea that mathematical knowledge is useful and connected	The development of a basic repertoire helps understand that mathematical knowledge is useful and connected	<i>No direct link with knowledge about mathematics</i>

# CHAPTER 6

## METHODOLOGY

## 6 Methodology

The first part of this dissertation deals with theoretical issues. In Chapter 2, I analysed two Mozambican didactic institutions' relation to limits of functions. In Chapter 3, I elaborated a new framework for *Mathematics for Teaching* (MfT) based on the anthropological point of view developed by Chevallard. In Chapter 4, I applied this framework to analyse MfT limits and show how the Mozambican secondary school relation to limits could be expanded. Finally, in Chapter 5, I focused on issues related to the design of a new institution where teachers could develop a new personal relation to limits, through a research group.

The empirical part of this study, the setting up of the new institution, selecting teachers willing to participate in this program, organising the collection and analysis of data in order to look at the evolution of these teachers' personal relation to limits during the research process, is discussed in chapters 6 to 11.

In this chapter I focus on methodological issues relating to the creation of the new institution, and to data collection and analysis. I also discuss issues of validity and ethics in this research. This chapter includes the following sections:

- 6.1. The research methods
- 6.2. Selection of teachers for the research group
- 6.3. Ethical issues
- 6.4. Data collection
- 6.5. Data analysis
- 6.6. Validity

### 6.1 The research methods

The literature about methodology in educational research usually presents two opposing models: the positivist, scientific model, and the interpretative, ethnographic model. According to the first approach, the “researchers, described as ‘positivists’, argue that social research must use methods and procedures of the natural or physics sciences” (Hitchcock & Hughes, 1989: 12). As a result, they

use quantitative methods. The researchers within the second model argue “for the importance of discovering the meanings and interpretations of events actors themselves have” (1989: 12). Therefore, they use qualitative methods.

These two conceptions correspond to two different ways of looking at social reality. Cohen & Manion (1994) examine the basic assumptions underlying these two opposing models in terms of contrasting ontologies, epistemologies and models of human being.

*Ontology* refers to issues concerning the nature of the social phenomena being investigated. Cohen & Manion (1994) speak about the “nominalist-realist” debate. They explain that positivists hold a realist position, which “contends that objects have an independent existence and are not dependent on the knower” (1994: 6). Social reality is “external to the individual and exists independently of individuals’ construct of it” (Hitchcock & Hughes, 1989: 17). In contrast, from an interpretative point of view, nominalist researchers consider that “objects of thoughts are merely words and that there is no independently accessible thing constituting the meaning of a word” (Cohen & Manion, 1994: 6).

*Epistemology* concerns “the basis of knowledge, the form which it takes and the way in which knowledge may be communicated to others” (Hitchcock & Hughes, 1989: 14). Positivists view knowledge as “hard, objective and tangible” (Cohen & Manion, 1994: 6). As a result they see the researcher as an observer using the methods of natural science to describe the world. On the contrary, within the interpretative model, the knowledge is seen “as personal, subjective and unique”, which “imposes on researchers an involvement with their subjects and a rejection of the ways of the natural scientist” (Cohen & Manion, 1994: 6).

The third basic difference between the two sets of assumptions concerns the nature of human beings and, in particular, their relationship with their environment. Positivists see a human being as “responding mechanically to his environment” and contend that “human behaviour is governed by general laws and characterised by underlying regularities” (Hitchcock & Hughes, 1989: 27). Opposing that, the interpretative model states that human beings are “thinking, feeling, conscious, language- and symbol-using creatures” who are “capable of

choice and have the ability to act upon the world and to change it in line with their own needs, aspirations, or perceptions” (Hitchcock & Hughes, 1989: 28-29).

The basic differences between these two conceptions of social reality shape the kind of methods used by the researcher. Positivists will use methods and procedures of the natural sciences: they will rather apply quantitative methods. Within the interpretative model, where the principal concern is to understand how individuals create and interpret the world in which they are inserted, researchers prefer qualitative methods, which allow them to deepen their analysis.

The aim of this study is to investigate how teachers’ personal relation to a mathematical concept could evolve through their participation in a research group. I consider that the concept “limits of functions” does not exist independently of individuals, or groups of individuals (institutions), as explained in the presentation of Chevallard’s theory (see Chapter 2). Each individual who has been in contact with limits of functions through several institutions has a personal relation to this concept, and this relation can only evolve through a contact with another institution. I also consider teachers as human beings capable of choice and who are able to act upon the world and try to change it according to their own feelings and aspirations. I want to study the evolution of teachers’ personal relation to limits through their participation in the new institution that I created (see Chapter 5). This interpretative conception led my research to qualitative methods.

Periodic seminars were held where the teachers shared the progress of their research, the difficulties they were facing and their findings. These periodic seminars had two main objectives. The first one was to help teachers in their personal research, getting feedback and suggestions from the group. The second objective was to broaden their knowledge of MfT limits of functions. In fact, each teacher had his own research topic and, through this topic, mainly developed one or more aspects of MfT limits in schools. Their participation in the seminars allowed them to discuss other aspects of this concept with their colleagues and not be restricted by their own research topic.

Furthermore, to deepen the analysis of teachers' personal relation to limit, they were interviewed three times: at the beginning, in the middle and at the end of the research process.

I will explain in detail how I collected data in section 6.3. In the next section I explain how I selected teachers for my research group.

## **6.2 Selection of teachers for the research group**

The selection of teachers for the research group was guided by two main concerns: answering my research question and ethical issues.

To answer my research question, I needed to form a group of teachers researching different aspects of the limit concept, and to observe them through the research process. Ideally I wanted to work with four teachers, but chose to work with six, because some of them could drop out during the process. As a matter of fact one of them dropped out after the first seminar, and another one passed away before he concluded his dissertation.

Another concern when choosing the teachers was ethical. As I already explained (see Chapter 5) I wanted the teachers to have their own agenda when entering the group. Secondary school teachers are usually very busy in Mozambique, often teaching in more than one school, for example in a public school during a part of the day and in a private school during another part. They probably would not be willing to spend time in research without any personal benefit. I then decided to form a group of student- teachers from the Pedagogical University (PU) or from the Masters Degree at the Faculty of Education at Eduardo Mondlane University (EMU). All of them needed to write a dissertation to conclude their studies and I could help them by being their supervisor.

Considering that a dissertation at Masters' level is more challenging than one at Honours level, and that some of my topics were not suitable for this kind of dissertation (as for example the history of the limit concept), I first had a meeting with all Masters Degree students in Mathematics Education at the Faculty of Education to explain the aims and organization of my research and ask for volunteers (see "Letter to teachers", Appendix 6.1). Two students showed interest

in participating in my project and filled the form that I had prepared for this purpose (see Appendix 6.2).

I then had a meeting with final year students of the Pedagogical University (PU) who were waiting for a supervisor to conclude their teacher training. As for the students of the Faculty of Education, I explained my research and gave them forms to be filled in and collected by one of their lecturers. To these students I only presented four possible topics to choose from. I received nine applications, and had a meeting with my colleague from PU to select some students, considering their first choice and my colleague's opinion on their reliability. I was worried that some of the student-teachers would drop out of the research group before concluding their dissertation. In fact, one of the four PU teachers that I selected dropped out before our first meeting. He was immediately substituted by another teacher who had indicated the same topic as his first choice.

In that way, at the beginning of the study, six teachers were involved in the group: two teachers taking a Masters Degree at the EMU Faculty of Education, and four student-teachers in their final year at PU. All of them were using their research for their dissertation. By this means, all teachers participating in the research group had their own agenda for joining the group.

Having described how I selected the teachers for my research, I will now explain how I dealt with other ethical issues during this study.

### **6.3 Ethical Issues**

Qualitative research brings up ethical questions. Deyhle, Hess & LeCompte (1992), quoted by Miles & Huberman (1994), suggest that these ethical issues can be nested in five general theories:

- A *theological* theory, which judges actions according to primary ends, good in themselves;
- A *utilitarian* pragmatic approach that judges actions according to their specific consequences for various audiences;
- A *deontological* view which invokes one or more universal rule;
- A *critical* approach, which judges actions according to whether one provides



benefits to the researched individual and/or becomes an advocate for them;

- A *covenantal* view, which judges actions according to whether they are congruent with specific agreements made with others in a trusted relationship.

In this study, I deal with ethical issues in a utilitarian critical approach, trying to avoid any embarrassment that the participants could feel during the research process and analysing the direct benefits that they could find in it. For this purpose, I tried to answer some of the questions that Miles & Huberman (1994: 291-296) consider as issues that typically need attention in qualitative research.

### ***Worthiness of the project***

*“Is my contemplated study worth doing? Will it contribute in some significant way to a domain broader than my funding, my publication opportunities, and my career?”*

With this research, I intended to contribute to the field of Mathematics Education in several ways. In the first place, the analysis of Even’s framework for Subject Matter Knowledge from an anthropological point of view led me to elaborate a more systematic framework for the study of MfT. These results can be useful for both pre-service and in-service teacher training, not only in Mozambique but also in other countries.

In the second place, my analysis of how teachers learnt through research would also be a contribution for the field of mathematics education, particularly in the debate about *action research* or *teachers as researchers*.

### ***Informed consent, benefits, costs, and reciprocity***

*Do the people I am studying have full information about what the study will involve? Is “their” consent to participate freely given? Does a hierarchy of consent affect such decisions? What will each party to the study gain from having taken part?*

All teachers involved in the research group were informed about the aims of this study and its methodology, and volunteered to join the research group. As student-teachers, they also had their own agenda, their dissertation. To give them some freedom in choosing their subject, I asked them to fill in a form where they had to

indicate three choices from my list of topics for their research. I tried to give their first choice to all of them.

***Privacy, confidentiality and anonymity***

*In what ways will the study intrude, approximate people more than they want?  
How will information be guarded? How identifiable are the individuals and organisations studied?*

This issue raised some problems in this research. On the one hand, as I already explained, from the six teachers who initially joined the group, two were EMU Masters Degree students and four were PU Honours students. Furthermore, five were male and only one female. The indication of these particularities in my dissertation could facilitate the identification of some of them. In order to protect all teachers' identity, I therefore decided to distinguish neither between PU Honours students and EMU Masters Degree students, nor between males and female. I would speak about the teachers as if they were all male PU Honours students. However, even trying to protect teachers' anonymity, in a small circle as mathematics teachers and student mathematics teachers in Maputo, it is rather difficult to avoid recognition of individuals. It is easy to know which teachers or student teachers worked with me, and even changing the name of the participants in my reports, they would probably be easily recognisable.

On the other hand, I was not sure whether the participants wanted anonymity or not. They might consider that they gain some prestige with their participation in the research group and might want their colleagues to know that they were involved in it. In fact when I raised the issue of anonymity during a seminar, some of the teachers said that they did not mind having their real name in my dissertation. As there was no consensus on this, and that giving some of the names would facilitate the identification of other participants, I changed all teachers' names. From this point onward, I will refer to these teachers by their pseudonyms. The list for the teachers involved in the group, designated by their pseudonyms, and the topic of their research is indicated in Table 6.1.

**Table 6-1 Teachers and research topics**

<b>Teacher</b>	<b>Research topic</b>
Abel	Alternative ways of introducing the limit concept in schools
Bernardo	Basic repertoire for teaching limits of functions in schools
David	Applications of the limit concept in mathematics and in other sciences
Ernesto	Students' conceptions and difficulties when learning the limit concept
Frederico	The history of the limit concept and its implications for teaching
Mateus	The use of different settings and models to teach the limit concept

## 6.4 Data collection

During the whole process I collected the following data: Three interviews with the four teachers who completed their dissertation (one with Bernardo, who dropped out after the first interview; two with Ernesto, who passed away before the third interview), 13 seminars, several versions of each teacher's dissertation, notes taken during the individual supervision sessions and my journal. Table 6.2 provides a summary of data collected (see next page).

Below is an explanation of the process involved with each of these components.

### 6.4.1 Interviews

Kvale (1996) describes the research interview as “an interview whose purpose is to obtain descriptions of the life world of the interviewee with respect to interpreting the meaning of described phenomena” (1996: 5-6).

Interviews of the teachers seemed to be an appropriate way of interpreting what the limit concept meant for them. Considering that I also wanted to analyse the evolution of their personal relation throughout the research process, I decided to hold three individual interviews with each teacher: one at the beginning of our work together, one during this process and the last one at the end.

**Table 6-2 Summary of data collected**

<b>Type of data</b>	<b>Nature of instrument</b>	<b>Period collected</b>	<b>Teachers involved</b>
<b>1<sup>st</sup> interview</b>	Semi-structured interviews, approximately two hours. Audio-taped and transcribed.	Aug 02 - Oct 02	All six teachers
<b>Supervision sessions</b>	Teachers writing with my comments, notes, my journal.	Sept 02-April 02	All, except Bernardo after Nov 02
<b>Seminars 1-3</b>	Discussion of works in progress (Seminars 1-2) $\epsilon$ - $\delta$ definition (Seminar 3). Audio and video-taped and transcribed.	Nov 02 - Apr 03	All, except Bernardo after Seminar 1
<b>2<sup>nd</sup> interview</b>	Semi-structured interviews, approximately two hours. Audio-taped and transcribed.	April 03	Five teachers
<b>Supervision sessions</b>	Teachers writing with my comments, notes, my journal.	Apr 03-Dec 03	Five teachers
<b>Seminars 4-10</b>	Discussion of works in progress (Seminars 4, 6-10); Different settings and registers (Seminar 5). Audio and video-taped and transcribed.	May 03 - June 04	Five teachers (except Ernesto after Seminar 8)
<b>Presentation</b>	Last version of dissertation. Presentation and answer to jury questions. Audio and video-taped and transcribed.	Dec 02	Mateus, Frederico
<b>Seminar 11-12</b>	Discussion of works in progress. Audio and video-taped and transcribed.	Feb 03-Mar 03	Four teachers
<b>3<sup>rd</sup> interview</b>	Semi-structured interviews, approximately two hours. Audio-taped and transcribed.	Mar 03	Mateus, Frederico
<b>Supervision sessions</b>	Teachers writing with my comments, notes, my journal.	Mar 03-June 04	Abel, David
<b>Seminar 13</b>	Discussion of works in progress. Audio and video-taped and transcribed.	April 04	Four teachers
<b>3<sup>rd</sup> interview</b>	Semi-structured interviews, approximately two hours. Audio-taped and transcribed.	Sept - Dec 2004	David, Abel
<b>Presentation</b>	Last version of dissertation. Presentation and answer to jury questions. Audio and video-taped and transcribed.	Nov 04	David
<b>Supervision sessions</b>	Teachers writing with my comments, notes, my journal.	July 04-May 05	Abel
<b>Presentation</b>	Last version of dissertation. Presentation and answer to jury questions. Audio and video-taped and transcribed.	Sept 05	Abel

***First interview***

In order to analyse the evolution of the teachers' personal relation to limits of functions during the research process, I needed to examine carefully their relation to this concept before they joined the group. They had been in contact with limits through different institutions, or through the same institution but at different times or holding a different position (student or teacher). Because they each had different histories with respect to their contact to the limit concept, their personal relation could differ.

I chose to construct their prior relation to limits of functions from an interview for two main reasons. Firstly, an individual interview would allow an in-depth analysis of the teachers' knowledge about limits of function as well as their ideas about teaching this topic in Mozambican secondary schools prior to the research process. It would allow me to ask more questions, to probe some points deeply, and see what strategy they used to solve certain tasks. A questionnaire would not provide as much information about these issues as an interview. In the second place, I wanted to create a personal relationship with each teacher before he joined the group. We were going to work together over at least one year and my wish was to create a good relationship with each of them at an early stage. I did not want them to consider this first contact as trying to test their knowledge about limits of functions, but as a conversation about this concept. A questionnaire would be more impersonal and could be regarded as an evaluation test.

For these reasons I conducted semi-structured interviews focusing on the one hand on the history of teachers' contact with limits through the several institutions where they had met this concept and, on the other hand, on their personal ideas about the teaching and learning of limits of functions at secondary school level.

Two pilot interviews were conducted: the first one with a PU student who never had taught limits and was not willing to join the group, the second one with a PU lecturer who was also teaching limits in a private Mozambican school. These two pilot interviews allowed me improve the guide for the first interview (see Appendix 6.3) and get feedback from the interviewees about their feelings during the interview.

The first interviews with the teachers involved in my research group took place from August to October 2002 (see Timetable in Appendix 6.4).

### ***Second interview***

The second interview took place in April 2003, more or less five months after the first seminar. The aims of this interview were to explore the teachers' feelings about the following questions:

- How had their knowledge and conceptions of limits of functions and its teaching in secondary schools evolved since the beginning of the research?
- What role had their personal research, the supervision sessions, the seminars, and the first interview played in this evolution?

The teachers had been informed about the aims of the interview during the previous seminar (third seminar), where I suggested that they could prepare themselves for this interview and bring any documents they wanted to use or show me.

This interview was less structured than the first one (see Guidelines in Appendix 6.5).

### ***Third interview***

The third interview was held at the end of each teacher's personal research, and consequently at different times. Mateus and Frederico concluded and defended their dissertation in December 2003, and were interviewed for the third time in March 2004. In the meantime, Abel and David were still working on their research, and we had two more seminars (January and February 2004) dedicated to the discussion of the progress of their work. As I detected during the interviews that Mateus and Frederico had difficulties working within the graphical register, I dedicated the 13<sup>th</sup> and last seminar (April 2004) to this issue. My intention was to have a discussion with Mateus and Frederico afterwards, to complete the information gathered during the third interview. This was unfortunately not possible because Frederico was sent to teach in another province.

David completed his dissertation in June 2004, but the presentation of his work only took place in December. I interviewed him in September, before this

presentation. Abel was almost finished with his dissertation in December 2004. I decided to hold the interview before he was completely finished and had defended his dissertation because I was running late with my data collection. In fact he only defended his dissertation in September 2005. As a consequence, the four teachers were not exactly in the same conditions for the third interview (see Timetable in Appendix 6.4).

The aims of the 3<sup>rd</sup> interview were to:

- Find out how teachers' knowledge about limits of functions evolved since the beginning of the research;
- Analyse the evolution of their ideas about the teaching of limits in Mozambican secondary schools;
- Analyse the role of each activity in the evolution of their knowledge and ideas: their personal research, the supervision sessions, the seminars, the first and the second interviews.

This analysis of their knowledge was done considering the five research topics:

- History of the limit concept;
- Several settings and registers in which limits can be studied;
- Different ways of approaching the concept;
- Applications of the limit concept in Mathematics and in other sciences;
- Students' conceptions and difficulties when learning limits.

To analyse teachers' ideas about teaching limits, I used the same tasks as during the first interview, in order to compare their opinion about these tasks at the end of their research with those at the start of the research process.

We spoke about the 3<sup>rd</sup> interview during the 11<sup>th</sup> seminar (21 of February 2004). As for the second interview, I explained the aims of the interview, and told them that they could bring to the interview any documents they wanted to show me. This interview was also semi-structured (see Guidelines in Appendix 6.6).

### 6.4.2 Seminars

During each of periodic seminars, the teachers presented their work, in terms of intentions, findings, or difficulties, and discussed them with the whole group. Some seminars were also dedicated to discussing special aspects of limits. In the 3<sup>rd</sup> seminar, we discussed the  $\varepsilon$ - $\delta$  definition; in the 5<sup>th</sup> seminar, we discussed the different settings and registers in which limits of functions can be studied; and in the 13<sup>th</sup> seminar, we spoke about sketching a graph using the limits of the function (see the content of the seminars in Appendix 6.7).

All seminars were both audio-taped and video-taped.

### 6.4.3 Other data

#### *Dissertations*

While the teachers were writing their dissertation, several drafts were produced. I commented on these drafts and discussed them with each teacher during the supervision sessions or during a seminar. I kept a copy of all commented drafts.

#### *Dissertation presentations*

Each teacher had to present his dissertation and answer questions asked by a jury. As their supervisor I was part of the jury. As for the seminars, I both audio-taped and video-taped these presentations.

#### *Journal*

In this research I played a double role. On the one hand, I was the teachers' supervisor in their personal research. On the other hand, I was myself a researcher of teachers' personal relation to limits of functions. As a researcher I had to observe the teachers and take notes about the evolution of their relation to this concept. During the individual sessions with each teacher, my main role was to supervise their work, and during the seminars and the interviews, my main role was to observe how their personal relation to limits evolved through the interactions within the group. Nevertheless the teachers' personal relation to limits could also evolve during their individual work, and I had to take notes about that during the supervision sessions. In a similar way, I sometimes had to act as a



supervisor during the seminars. In order to analyse the difficulties of being both a supervisor and a researcher at the same time, I decided to write a journal where I would take note of the problems faced, the solutions found, and comments about them during this process.

In fact, this journal played a different role in this research. I used it to take notes of what occurred during the supervision sessions and the seminars, and about my feelings during the research process. I wrote a few observations about my double role, which I will refer to in the last chapter.

### ***Overview***

The data collection lasted longer than I expected (from August 2002 to September 2005). At the end of this data collection, I had three audio-taped interviews with each of the four teachers who concluded their dissertation, thirteen audio and video-taped seminars, several drafts of each teacher's writing for their dissertation, four audio and video-taped presentations of their final dissertation, and my journal. I then had to select the data to be analysed in depth to answer my research question. The next section explains how I selected and analysed the data.

## **6.5 Data-analysis**

Data-analysis is a very delicate task in qualitative research. Issues of reliability and validity of claims coming from qualitative data have been stressed by several authors (Kvale, 1996; Maxwell, 1992; Yin, 1994).

As mentioned in the previous section, I had a lot of data to analyse. Miles & Huberman advise on the risk of data overload in qualitative research. They

strongly recommend early analysis, in order to organize the data. It helps the field worker to cycle back and forth between thinking about the existing data and generating strategies for collecting new, often better, data. It can be a healthy corrective for built-in blind spots. It makes analysis an ongoing, lively enterprise that contributes to the energizing process of fieldwork (1994: 50).

Following this advice, I began to transcribe interviews and seminars during the data collection period. This helped to detect some weak points in the teachers' mathematical knowledge of limits of functions. This is the case, for example, with

the formal definition. I realised during the first interview that all teachers had difficulties in understanding it. Consequently I dedicated a seminar to discussing this issue.

Data analysis aims to answer the research question. In this case the aim was to analyse the evolution of teachers' personal relation to limits of functions through the research process. The following sub-questions were set to aid achievement of the main objective:

- What was each teacher's personal relation to this concept prior to the research process?
- What relation did they hold at the end of the research?
- What evolved, what did not? How and why did this evolution/non-evolution take place?
- What was the role of each institution in the evolution of their relation to limits?

To answer these questions, I used as a starting point the first and third interviews. They were supposed to provide most of the information both about teachers' initial relation to limits (in the first interviews and in some teachers' utterances during the third) and about their personal relation at the end of the research process (third interview). This first image of the evolution of teachers' personal relation was then to be checked and refined by further data analysis.

Step by step data analysis is now presented in more detail.

### **6.5.1 Interview analysis**

All interviews were audio-taped. During the interviews I also took notes, particularly during the analysis of the tasks presented to teachers, in order to be sure when specific utterances were made. The analysis of interviews included the following steps: transcribing, coding, restructuring, drawing comparison tables, and saturating the data.

### ***Transcription***

All interviews took place in a quiet room at my department. As a consequence the audio-tapping was of good quality. I transcribed the interviews myself, going back and forth over the same passage until I fully understood it. Once the interview was transcribed, I checked the accuracy of this transcription by listening to the interview once more, while comparing it to my transcript. This enabled me to correct certain misunderstandings that sometimes caused a loss of meaning in the context of specific utterances of small parts of the interview.

The interviews were transcribed word by word, including repetitions and hesitations, from the audio-tapes in Portuguese, and given the form indicated in the following quote.

223 A: (...) Foram dois, dois anos  
 224  
 225 I: Hum, hum  
 226  
 227 A: 82, 83,  
 228  
 229 I: Ok  
 230  
 231 A: Posto isso ...

The first column gives the line numbers. A free line was given between two interventions for a lighter presentation and to facilitate the codification of the data. The speaker is indicated by a letter: A, D, E, F and M for the teachers (respectively Abel, David, Ernesto, Frederico and Mateus) and I for the interviewer.

In these transcriptions the following conventions were used:

...	small pause
[silence]	long pause
[extended silence]	longer pause
[looking at Sheet 1]	other useful information for the situation, or indicating emotional expressions such as laughter and sighing, has been indicated in square brackets, as in the example.

The duration of pauses in the teachers' speech were not quantified because they were not relevant for the research.

### ***Coding***

Miles & Huberman consider that “codes are efficient for data-labelling and data-retrieval. They empower and speed up analysis.” (1994: 65). I started a list of codes prior to the fieldwork. These codes were created according to the aspects I wanted to observe, as defined on my framework of MfT limits of functions. Each code was chosen semantically close to the term that it represents and was operationally defined. During the coding process, other codes were introduced to classify some of the teachers’ utterances. For example, each interviewee spoke about him/herself, as a teacher, as well as a researcher, at some point during the interviews. These were also included in the initial list of codes. At times the teachers digressed and began speaking about some of their experience or giving their opinion on issues not really related to limits of functions. I classified these as “Other issues”. The final list of codes is presented in Appendix 6.8.

### ***Restructuring***

As the interview was semi-structured, the issues were not taken in the same order in each interview. Some items also appeared to be focused on twice or more during the interview. To facilitate the analysis, I reorganised the transcripts using the framework presented in Appendix 6.9.

The first interviews of Abel, Frederico and Mateus were analysed in detail, dividing the transcript according to this framework, and comments on the teachers’ utterances added. In some of the analyses I translated the quotes to enable my supervisors to verify my analysis. I always presented the Portuguese transcript with the English translation, as shown below.

M: (...) menos infinito [alínea a] ... já algébrico, não é?	M: (...) minus infinite [limit a] ... now it is algebraic, isn't it?
I: Ya	I: Yes
M: Então isto [alínea b], seria igual a um, isso ... também [alínea c] seria ... igual a 1 positivo...	M: Then this one [limit b], would be one, that one ... too [limit c] would be ... one positive ...
I: Euh, espera. Aqui o $x$ tende para aqui [mostrando -5] (I1/M/1337-44)	I: Er, wait a moment. Here $x$ goes to this [showing -5]

The first column shows the Portuguese version and the corresponding lines in the transcript: I1 for 1<sup>st</sup> Interview, M for Mateus, and 1337-44 for the lines in the transcript. The second column provides the English translation.

In the quotations, omission points indicated in brackets (...) mean that a part of the speech has been omitted.

In other cases, when the quotations were not considered so illuminating, a summary of the contents was made indicating the lines, as for example:

He said that he did not use the Grade 11 Mozambican textbook but saw it at the Nautical School library (I1/M/494-504).

After analysing three interviews (Abel, Mateus and Frederico) using this method, I analysed the other two initial interviews (David and Ernesto) in the same way but without translating all the quotes. Later on I decided not to use the results from Ernesto in my study because I had insufficient data from him<sup>13</sup>.

### ***Drawing comparison tables***

For the first interview, I drew up tables to compare the four teachers' knowledge about a specific aspect of MfT limits, using a summary of the teachers' comments about that aspect (see Appendix 6.10). I completed these tables when analysing the third interview, adding a summary of the teachers' assertions in each column. This enabled me to compare teachers' comments side by side (comparison between teachers at the same time) and vertically (same teacher's statements at different times) (see Appendix 6.11).

I drew my conclusions on the evolution of teachers' knowledge related to a special aspect from the respective comparison table, firstly teacher by teacher, and then going back to the analysis of each interview or even to the transcript when I needed to check a certain utterance. I then made the comparison between teachers, identifying similarities and differences, and trying to explain them. I also took note of other parts of the data that I considered important to add to the information about this specific issue.

I drew comparison tables for all of the following aspects of MfT limits of functions: the essential features, the  $\varepsilon$ - $\delta$  definition, the strength of the concept, the first encounter with the limit concept, the graphical register, the numerical

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<sup>13</sup> Ernesto passed away in November 2003, without concluding his dissertation and being interviewed for the third time

register, the algebraic setting, the natural language register, the basic repertoire, knowledge about mathematics, and students' conceptions and difficulties.

### ***Saturating the data***

The saturation of data was done by re-reading all interviews to ensure that all teachers' utterances regarding limits of functions were considered for the analysis of at least one of the aspects of MfT limits. This was also done with the seminars as well as my journal.

### ***Selecting aspects of Mathematics for Teaching limits for in-depth analysis***

According to my framework, MfT has many components. In fact, due to constraints of the institution that I built, I did not collect systematic data about all of these aspects. These constraints were:

- The number of teachers involved in the research group. As explained before, the group began with six teachers, but two of them were lost during the process. These teachers were researching “Basic repertoire” and “Students' conceptions and difficulties when learning the limit concept”. Obviously the loss of two participants impoverished the learning of these two aspects of MfT limits, as well as the discussion about the other aspects.
- Some aspects of MfT limits were directly related to one of the teachers' research topics, for example “The first encounter” and “Social justification”, some appeared across several topics, for example “Different representations”, and others were implicit through topics, but did not correspond to any specific topic, as for example “Essential features”. Consequently, data collected about all these aspects were not exactly of the same type.
- I wanted the teachers to feel comfortable within a group of colleagues, and considered my role as a facilitator more than as a lecturer. I also wanted to deepen the analysis of the aspects of MfT limits which were less usual for these teachers, according to my prior analysis of the institutional relation of Mozambican secondary schools. For these reasons I conducted semi-

structured interviews and chose not to use a questionnaire, which could have been more systematic but less productive in terms of my relation with teachers and an in-depth discussions of certain questions.

- As in any didactic institution, time was also a constraint. The seminars and the interviews were an extra activity for these teachers, who were already very busy with their studies and their teaching. For this reason I wanted seminars and interviews to last less than two hours. This meant that an in-depth discussion about a particular aspect restrained data collection about other aspects of knowledge. In particular, this was the case of the third interviews, where a lot of time was spent with graphical tasks.

Given these constraints, and looking at the general analysis of all teachers' interviews, I selected five of these categories or sub-categories of MfT limits for a deeper analysis. These were:

- The essential features of the concept;
- The  $\varepsilon$ - $\delta$  definition;
- The social justification for teaching limits at secondary school;
- The organization of the first encounter with the limit concept;
- The graphical register.

I will now explain the reasoning behind the selection of these categories or sub-categories, and state why I excluded the others.

### ***Aspects of MfT limits selected for this study***

**Essential features** (sub-category of scholarly mathematical knowledge) - The study of the institutional relation of Mozambican secondary schools to the limit concept showed that it was mainly considered from an operational point of view (see Chapter 2). As expected, data analysis showed that teachers' personal relation to limits prior to the research was also dominated by this feature. It also showed that their personal relation to limits with respect to this aspect evolved during the research process for all four teachers. It is for these reasons that this aspect was included in this study.

**$\epsilon$ - $\delta$  definition** (also part of scholarly mathematical knowledge) - The study of the institutional relation of Mozambican secondary schools to the limit concept showed that the  $\epsilon$ - $\delta$  definition was taught but not used in practice (see Chapter 2). The same applied to the Pedagogical University. Based on this, I presumed that student-teachers might conclude that to define a concept is only formal, and could not consider the definition as a reference for the concept. In fact data analysis showed that none of the teachers had understood the  $\epsilon$ - $\delta$  definition at the beginning of the research. It also showed that their knowledge of this definition did not evolve much during the research process, which is why this aspect was also selected for this study.

**Social justification for teaching the limit concept** - At the beginning of the research process, all four teachers seemed to have a poor understanding of the reasons why limits were taught in schools. “Applications of limits in mathematics and in other sciences” was the topic of David’s dissertation and therefore a discussion point during several seminars. Through these discussions, all teachers became aware of the importance of the limit concept. I therefore considered this category worth including in this study.

**The first encounter** - The way limits of function are taught in Mozambican secondary schools and at universities does not give teachers much information about different ways of organizing the students’ first encounter with limits. As expected, at the beginning of the research process, they only knew the traditional way of teaching limits in schools. “Alternative ways of introducing limits in schools” was the topic of Abel’s dissertation and was also discussed in several seminars. Teachers’ knowledge about this aspect appeared to evolve substantially during the research process. For these reasons, this was an important aspect to be included in this study.

**Graphical register** (sub-category of the practical block) - The graphical register is hardly used to study limits in schools (see Chapter 2). As expected, the teachers were not accustomed to graphical tasks involving limits. Furthermore, when trying to solve some of these tasks, they faced difficulties that I had not expected, and some of these difficulties remained up until the end of the research process.



This aspect strongly related to David's topic, with the applications of limits in graphs, and to Mateus' topic "Different settings and registers". For these reasons, this was also an important aspect to be included on this study.

### ***Aspects not selected***

Other categories or sub-categories of MfT limits were not used in this study, either because data about them was insufficient or because the results were not as rich as for the aspects selected.

**Theorems and proof about limits** (part of the scholarly mathematical knowledge) - Theorems and proof about limits are not part of the secondary school syllabus. For this reason this issue was not a focus of the interviews or seminars.

**Algebraic setting** (part of the practical block) - As shown in Chapter 2, the Mozambican secondary school relation to limits is mainly algebraic. It was to be expected that teachers would not have many difficulties in calculating limits this way. For this reason, I did not dedicate much time to this aspect of limits during the interviews.

**Numerical register** (also part of the practical block) - The teachers did not show difficulties with this register. It was not a focus of the interviews.

**Linguistic register** (also part of the practical block)- During the interviews, all teachers correctly used the language related to limits.

**Students' conceptions and difficulties** - Unfortunately Ernesto, who was researching this topic, passed away before concluding his work. For this reason data about this important aspect of MfT limits was missing.

The fact that these categories or sub-categories of MfT were not included in this analysis does not mean that they are not important aspects of MfT limits. As shown before, MfT limits has many components, and choices had to be made about which components were to be focused on in this study for in-depth analysis. The aspects chosen were the most sensible in the context of this research. However, further study of other aspects is also important.

Having selected the aspects of MfT limits of functions that I wanted to focus on, I then looked for more information about all of them in other data, particularly the seminars and teachers' dissertations.

### 6.5.2 Seminars and dissertations

All seminars were transcribed from the tape-recordings. When necessary the video recording was used to complete the information, in particular for the 3<sup>rd</sup> seminar which took place in a very noisy room. Not all seminars were used for data analysis, because many parts of these seminars did not provide information about the evolution of teachers' knowledge. For example, time was spent discussing issues related to the writing of a dissertation, such as research methods, the use of a computer, and referencing. I only selected the parts of the seminars that provided information about the aspects of MfT limits selected for deeper data analysis: the 3<sup>rd</sup> seminar, as a source of information about teachers' knowledge of the  $\varepsilon$ - $\delta$  definition (Chapter 9), the 12<sup>th</sup> seminar for the first encounter (Chapter 7), and the 13<sup>th</sup> seminar for the graphical register (Chapter 8). Each chapter provides information for the choice of data and how it was used.

I also used parts of the teachers' dissertations related to these specific aspects of teachers' knowledge: Abel's dissertation for the first encounter (Chapter 7), and David's dissertation for the social justification for teaching the limit concept (Chapter 9). Other data was also used as background information to help me in the interpretation. My journal, which I read carefully, also helped me analyse my double role in this study, as a supervisor and as a researcher.

The data selected for analysis is summarized in Table 6.3.

**Table 6-3 Data selected for analysis**

First Encounter	1 <sup>st</sup> and 3 <sup>rd</sup> interviews, Abel's dissertation and 2 <sup>nd</sup> interview, 12 <sup>th</sup> seminar
Graphical Register	1 <sup>st</sup> and 3 <sup>rd</sup> interviews, David's dissertation, 13 <sup>th</sup> seminar
Social Justification	1 <sup>st</sup> and 3 <sup>rd</sup> interviews, and David's dissertation
Essential Features	1 <sup>st</sup> and 3 <sup>rd</sup> interviews
$\varepsilon$ - $\delta$ Definition	1 <sup>st</sup> , 2 <sup>nd</sup> and 3 <sup>rd</sup> interviews, 3 <sup>rd</sup> seminar

The next step, after selecting the data and analysing the evolution of teachers' personal relation to limits was, for all five aspects, to compare the evolution and present the results.

### **6.5.3 Structure of the analysis**

The analysis of the evolution of the teachers' personal relation to limits according to the five aspects of MfT limits selected for this study showed that their knowledge about three aspects evolved substantially (essential features, social justification, and the first encounter), but that they had faced many difficulties with the other two aspects (the  $\varepsilon$ - $\delta$  definition, and the graphical register).

Obviously these differences need to be explained. Hence in this thesis I have presented in the first place the categories where teachers learnt a lot during the research process, and then the categories where learning was more problematic. The chosen order is the following:

- (i) “The first encounter”, because this aspect also enables me to tell the teachers' story through their description of their several encounters with the limit concept. Furthermore, it provides information on teachers' ideas on how to teach limits in schools, which can also relate to other aspects of limits and therefore be potentially useful in further chapters.
- (ii) “The social justification” and “Essential features”, as they are the two other aspects where learning occurred smoothly.
- (iii) “The graphical register” was selected as the first aspect in which learning was restricted. I had a lot of data regarding this aspect. I would then be in a position to draw some hypotheses to explain the differences between teachers' learning on this aspect of MfT limits and the previous ones.
- (iv) “The  $\varepsilon$ - $\delta$  definition” to confirm my hypotheses generated through analysis of previous categories.

The first chapter of data analysis is presented in detail, both because it introduces the teachers and because it explains how I did the in-depth analysis of data. For

other aspects, the analysis has been done using the same procedures, but a shorter version of this analysis is presented.

Furthermore, for each of these aspects, I defined categories to structure the analysis of the evolution of teachers' knowledge. First, I defined categories emerging from data analysis for each aspect. However, when I brought them together and tried to find a thread between these categories, I realized that for four aspects (social justification, essential features, graphical register,  $\varepsilon$ - $\delta$  definition) my categories related more to mathematical knowledge, while the fifth one (first encounter) mixed mathematical knowledge and teaching ideas. For some of the former aspects, I had then added comments about teachers' ideas about teaching related to this aspect.

I realised that I was coming back to SMK and PCK, but in a different way. Instead of separating each aspect of MfT limits within the two categories SMK and PCK, I rather considered that each of the aspects of MfT limits defined in my framework had two components: mathematical and pedagogical. For certain aspects, one of these components could be stronger than the other, but they were deeply linked together.

I then came back to my data to track the missing component, defining two kinds of categories for each aspect:

- categories related to mathematical knowledge, ranking from “knowing less” to “knowing more”;
- categories for teachers' ideas about teaching, related to this aspect of limits, ranking in several degrees, from “being close to the secondary school institutional relation to limits” to a “challenging this institutional relation”.

The number of categories depends on the nature of the aspect (more mathematical or more pedagogical) and of the amount of data collected. A summary of the categories can be found in Appendix 6.12.

## 6.6 Validity

The nature of validity differs in qualitative research and quantitative research. Quantitative researchers use statistical models to analyse the validity of their findings, while qualitative researchers usually consider understanding as a more fundamental concept than validity. Maxwell (1992) presents a typology of five broad central validity categories for qualitative researchers, which is also a typology of the kinds of understanding at which qualitative research aims: descriptive validity, interpretive validity, theoretical validity, generalizability, and evaluative validity. I will use these categories to explain what kind of validity is to be expected in this study.

***Descriptive validity*** refers to “‘acts’ rather than ‘actions’ – activities seen as physical and behavioural events rather than in terms of the meanings that these have for the actor or others involved in the activity” (1992: 286).

In order to ensure a valid description of the events, statements and behaviours of the teachers involved in this project, I carefully transcribed all interviews and seminars, using audio-tapes and, when necessary, video-tape, and indicating any special feature in the speech relevant to its interpretation, as for example pauses or hesitations.

***Interpretive validity*** “seeks to comprehend phenomena not on the basis of the researcher’s perspective and categories, but from those of the participants in the situation studied” (1992: 289).

To reach interpretive validity, I tried to interpret the events, statements and behaviours of the teachers using their own language and concepts using my knowledge of mathematics teachers in secondary school acquired through my experience in pre-service and in-service training. The interpretation given by the participants themselves during the seminars and interviews also helped me acquire interpretative validity.

***Theoretical validity*** “refers to an account’s function as an explanation” of the phenomena, it “addresses the theoretical conclusions that the researcher brings to, or develops during the study” (1992: 291). Maxwell distinguishes two aspects of

theoretical validity: “the validity of the concepts themselves as they are applied to the phenomena, and the validity of the postulated relationships among the concepts” (1992: 291). Theoretical validity includes aspects of what is generally known as constructed validity and internal validity.

In this study, I interpreted teachers’ behaviours and utterances from an anthropological point of view (Chevallard, 1992). To ensure the theoretical validity of this interpretation, I first analysed the relation to limits of functions of the two main didactic institutions where teachers met this concept. I used all documents available in order to cross-validate information. I then referred to this analysis when analysing teachers’ personal relation to limits. For this analysis I also used several sources of evidence: interviews, seminars, supervision sessions, and dissertations. My interpretation of all these has been discussed with my supervisors and sometimes presented in seminars and conferences with other colleagues.

***Generalizability*** “refers to the extent to which one can extend the account of a particular situation or population to other persons, times, or settings than those directly studied” (Maxwell, 1992: 293). Many qualitative researchers distinguish between external generalizability and internal generalizability. For example, according to Maxwell (1992)

In qualitative research, there are two aspects of generalizability: generalizing within the community, group, or institution studied, to persons, events, and settings that were not directly observed or interviewed (internal generalization); and generalizing to other communities, groups or institutions (external generalization). Internal generalization is far more important for most qualitative researchers than is external generalization of their accounts (1992: 293).

By its anthropological approach, this study is grounded in the social context of some Mozambican didactic institutions and their relation to the limit concept. This cannot obviously be generalised to other countries, where the relation of didactic institutions to this concept may be very different. Even in Mozambique, certain institutions teach limits in a different way, as for example Kitabu College and the Institute for Transport and Communication (ITC).

As Maxwell argues:

Qualitative studies are usually not designed to allow systematic generalizations to some wider population. Generalization in qualitative research usually takes place through the development of a theory that not only makes sense of the particular persons or situations studied, but also shows how the same process, in different situations, can lead to different results (1992: 293).

In this research, I worked with specific teachers, and in specific circumstances. This research would most probably be different if done with other teachers or under other circumstances. For this reason, I do not claim that the findings reached from this study of four teachers can be generalized to the whole community of pre-service or in-service teachers at the same level in Mozambique. Furthermore, it is not the primary concern of this research. My main objective is not to generalise my accounts to other teachers, but to analyse how, why and whether changes can occur in teachers' personal relation to a concept.

Graven (2002) argues that

All research findings are embedded in specific contexts and therefore are not directly transportable to other contexts. However, research findings can exemplify issues that should be explored in a range of contexts and can contribute towards the generation of a cohesive theory on teacher learning (2002: 137)

This is the case of this study. The descriptions of the evolution of MfT limits of the teachers involved in the research group cannot be generalised, as stated above. However these accounts can be used as a starting point for further research in other contexts.

***Evaluative validity*** is different from the previous types of validity in the sense that “it involves the application of an evaluative framework to the objects of study” (Maxwell, 1992: 295). The same author considers that “evaluative validity is not as central to qualitative research as are descriptive, interpretive, and theoretical validity” (1992: 295). This study did not deal with evaluative validity.

To sum up, in this research I dealt with the issue of validity in a pragmatic way, as far as possible in qualitative research. I kept my description close to the data, and my analysis close to the frameworks used and developed in the theoretical part of

this study. I also presented my results and discussed them with other mathematics educators.

This chapter has provided an explanation on how I selected and analysed data, taking into account ethical issues and the validity of this study. In the next chapters I present the results of this data analysis, using the sequence chosen as explained before: the first encounter (Chapter 7), the social justification for studying the limit concept (Chapter 8), the first encounter (Chapter 9), the graphical register (Chapter 10) and the  $\varepsilon$ - $\delta$  definition (Chapter 11).



# **CHAPTER 7**

## **THE FIRST ENCOUNTER WITH THE LIMIT CONCEPT**

## **7 The First Encounter with the Limit Concept**

Chevallard (1999) emphasises the importance of students' first encounter with a mathematical organisation (see page 73-74). In Chapter 2, I showed that in Mozambican didactic institutions, the first encounter with the limit concept was mainly organised through a “cultural-mimetic problematic”, particularly through sequences, the  $\epsilon$ - $\delta$  definition and algebraic tasks to calculate limits. As a consequence, Mozambican mathematics teachers would probably not be aware of other possibilities of introducing the limit concept in schools. Some of these possible ways of organising students' first encounter with the limit concept have been described in Chapter 4.

Abel's research topic, initially entitled “Alternatives ways of introducing the limit concept in schools” directly addressed the “first encounter” problematic. His work was presented and discussed in several seminars, in particular in the 12<sup>th</sup> seminar, where the teachers engaged in a discussion on how they would introduce this concept in schools.

This chapter provides an analysis of teachers' prior knowledge about different possibilities of organising students' first encounter with the limit concept at secondary school level, through their own experience of learning and, for some of them, of teaching this concept, as well as the evolution of this knowledge through the research project. The data analysed for this purpose comes from the four teachers' first and third interviews, Abel's dissertation and second interview, and the 12<sup>th</sup> seminar. This chapter is structured as follows:

- 7.1. Data collection and analysis
- 7.2. First interview
- 7.3. Abel's second interview and dissertation
- 7.4. 12<sup>th</sup> seminar
- 7.5. Third interview
- 7.6. Conclusions

## 7.1 Data collection and analysis

### 7.1.1 First interview

The first interview was my first personal contact with the teachers after joining the group and before they began their research. As I already explained in the methodological chapter (page 164), during the first part of the interview, I asked the teachers to try to rebuild the history of their relation to limits of functions through the several institutions where they met this concept. For each institution, I asked them the specific question: “Can you remember how the teacher introduced this concept?”, and all of them spoke about what they could recall about the way this concept was introduced in each of these institutions.

During the second part of the interview, I asked their personal opinion about the teaching and learning of limits of functions in secondary schools. As a support for this discussion, I showed them several definitions of limits, and several tasks using different representations: numerical, graphical, and algebraic (see Guidelines for teachers’ first interview, Appendix 6.3).

I also asked them the following questions specifically focussed on the first encounter:

- In Mozambican secondary schools, limits are usually introduced through sequences. What do you think about this way of introducing limits?
- What other ways of introducing limits do you think could be used in schools? Which one do you think would be more appropriate to secondary school level?

Using the transcripts of the first interviews, I took the answers to my specific questions about the first encounter during both parts of the interview, as well as other teachers’ utterances about this topic, as indicators of their prior knowledge about the way students’ first encounter with limits of functions could be organised.

I faced several difficulties when analysing the data.

Firsly, as the teachers had not been previously informed about the content of this interview, because I wanted to analyse their prior knowledge without any interference, it was difficult for them to recall how limits had been introduced in each institution, as stated by Frederico when speaking about his teacher training course:

Como foi há muito tempo ... talvez poderia, euh, lembrar que, como é que foi introduzido se tivesse recorrido por exemplo aos meus cadernos dessa altura, ver mais ou menos como é que foi introduzido mas, assim ... tal como foi introduzido sinceramente não, não posso precisar, não posso mentir (I1/F/202-205).

As it was a long time ago ... maybe I could, er, remember that, how it was introduced if I had looked at my exercise books for example, to see more or less how it was introduced but, like that ... how it was introduced sincerely I can't, I can't say, I can't lie.

As a consequence, some of the teachers' statements seem to contradict each other, as they tried to recall how the limit concept was introduced in a specific institution. I mention this fact when it is the case.

Secondly, it was sometimes difficult to distinguish in teachers' utterances parts related to the first encounter from parts related to the practical block, due to the dominance of the algebraic setting in Mozambican institutions. This is particularly the case of Abel as a secondary school teacher (see Section 7.2.1). I decided to include these parts of the practical block strongly related to the first encounter in this chapter because they appear to have a strong influence on the evolution of teachers' personal relation to the first encounter.

Finally, during the interview I showed the teachers several tasks, different from the tasks usually solved in secondary schools, such as numerical and graphical tasks. By doing so, I put them in contact with the limit concept through the new institution, and consequently through a new institutional relation to limits. This situation was clearly explained by David at the end of the interview:

Eu fiquei surpreendido com, com certas situações da, dos anexos que, que ia me mostrando, não é, porque eram variantes que eu nunca havia analisado assim ... então aquilo para mi foi uma surpresa. Mas eu, falando sério, eu gostei muito de, do ponto de vista porque são variantes que acredito que vão ajudar alguma coisa! (I1/D/1426-46)

I was surprised by, by some situations of, of the appendices that, that you showed me, you know, because, because they were variants that I never had analysed before ... then it was a surprise for me. But I, seriously speaking, I liked a lot the, this point of view because they are variants that I believe could be helpful!

At the end of the first interview some of the teachers (in particular Mateus and Frederico) were able to suggest another kind of approach to limits in schools, and these new ideas could have been influenced by the new kinds of tasks that I showed them. This means that, for these teachers, the evolution of their personal relation to the first encounter with limits already began during the first interview. In that way some teachers' ideas about teaching presented here are not exactly their previous ideas, but their views at the end of the interview. I mention this fact when it occurs.

### **7.1.2 Abel's dissertation**

Abel's topic, at the beginning of his research, was "Alternative ways of introducing the limit concept in schools". This topic evolved during his research. Besides a review of several ways of introducing the limit concept in secondary schools, he experimented with one of these in a classroom. His final dissertation was entitled "Introduction of the limit of a function concept using the graphical method with a computer" [Introdução do conceito de limite de uma função pelo método gráfico usando o computador.]. One section of the first chapter presented several ways of introducing the limit concept. In previous versions of Abel's work in progress, this part constituted a full chapter. This was presented and discussed during the supervision sessions and sometimes with the group during the seminars. In data analysis I mainly refer to the last version (August 2005), as well as the version discussed during the 12<sup>th</sup> seminar (March 2004).

In order to preserve the teachers' identity, I changed their name, as explained in the methodological chapter (see pages 161-162). For the same reason, I cannot provide the reference of Abel's dissertation.

### **7.1.3 Second interview**

The second interview took place in March 2004. At that time, we only had two seminars (see Timetable, Appendix 6.4). During the first two seminars, the teachers presented their topic, their ideas about what they were planning to do, and got feedback from the group. We also discussed forms of collaboration between the members of the group and common difficulties about the limit concept. At this early stage of the teachers' research, there was not much

discussion about the content of the dissertations. The third seminar was dedicated to one of the difficulties faced by teachers: the  $\epsilon$ - $\delta$  definition.

When I held the second interview, Abel already had some ideas about several ways of introducing the concept, but had not had an opportunity to discuss them in depth with his colleagues. As a consequence, the other teachers spoke very little about this issue during their second interview. For this reason, I only used the part of Abel's second interview related to the first encounter.

#### **7.1.4 12<sup>th</sup> seminar**

Parts of Abel's dissertation were discussed in several seminars (see Appendix 6.7). I chose to analyse the 12<sup>th</sup> seminar, which took place on 6 March 2004. During this seminar there was a focused discussion between the teachers on how they would introduce limits in schools, as we discussed Abel's chapter about different alternatives of introducing the limit concept.

Abel had circulated a copy of this chapter before the seminar, and the discussion had two parts. The first part was dedicated to discussing some aspects of the writing and of editing of their dissertations and not really to the teaching of the limit concept.

I then asked the question "Which alternative would you choose?" and the teachers engaged in a discussion about how they would introduce the limit concept in schools. During this second part of the discussion, the teachers revealed that they held different positions.

This seminar took place almost at the end of the research group: Frederico and Mateus had already defended their dissertations, David was almost finished, and it was the last seminar but one. I thus considered teachers' statements during this seminar as an indicator of their knowledge and ideas about how to organise students' first encounter with this concept at the end of the research process, the other spurce being the third interview.

#### **7.1.5 Third interview**

The third interview aimed to analyse the teachers' personal relation to limits at the end of the research process. As explained in the methodological chapter, this

interview was less structured than the previous ones (see Appendix 6.6). I asked the teachers to answer the general question “What do you think you have learnt since the beginning of the research?” focusing on each of the research topics. All four teachers spoke about Abel’s topic “Alternative ways of introducing the limit concept in schools”. I also asked them the following question: “If you had to teach limits at secondary school level, how would you organise this teaching?” The answers to this question also provided information about their knowledge of the different ways of organising students’ first encounter with the limit concept, and on how they were willing to use this knowledge in their teaching.

Frederico and Mateus were interviewed in March 2004, just after the 12<sup>th</sup> seminar. David’s interview took place in September and Abel’s in December 2004. Abel concluded his dissertation in August 2005.

#### **7.1.6 Analysing the evolution of teachers’ personal relation to the first encounter**

In order to analyse the teachers’ personal relation to limits with regard to the first encounter at the beginning and at the end of the research process, I analysed data from the four teachers’ first and third interviews, Abel’s dissertation and second interview, and the 12<sup>th</sup> seminar. In order to structure this analysis, I defined three categories related to their mathematical knowledge, designated by FE-MK1 to 2 (First Encounter – Mathematical Knowledge 1 to 2, Table 7.1), graduated from knowing less to knowing more, and six categories related to their ideas about how to organise students’ first encounter with limits, designated by FE-T1 to 6 (First Encounter – Teaching 1 to 6, Table 7.2, next page), graduated from aligning with the institution relation to developing a strong new personal relation.

**Table 7-1 Categories of teacher’s knowledge about the first encounter**

FE-MK1	The teacher only knows the way the first encounter with limits is organised in Mozambican secondary schools according to the syllabus.
FE-MK2	The teacher knows of other ways of organising the first encounter with limits of functions and is able to explain at least one of them.

**Table 7-2 Categories of teacher’s teaching ideas about the first encounter**

FE-T1	The teacher does not challenge the way the first encounter with limits of functions is organised in Mozambican secondary schools according to the syllabus.
FE-T2	The teacher does not challenge the way the first encounter with limits of functions is organised in Mozambican secondary schools but is aware of students’ difficulties.
FE-T3	The teacher knows the way the first encounter with limits of functions is organised in Mozambican secondary schools, is aware of students’ difficulties and suggests some changes to the institutional relation.
FE-T4	The teacher explains how he would organise the first encounter in schools in a different way.
FE-T5	The teacher explains how he would organise the first encounter within a new institutional relation and presents strong arguments to defend his ideas.
FE-T6	The teacher explains how he would organise the first encounter within a new institutional relation, presents strong arguments to defend his ideas, figures out possible problems and explains how to avoid them.

These categories emerged from the data analysis, and were used according to the following indicators.

FE-MK1 – The teacher is able to explain the way limits are usually taught in Mozambican secondary schools. I classified a teacher’s knowledge in this category when he described his contact with limits in terms of  $\varepsilon$ - $\delta$  definition, algebraic tasks, and sometimes sequences, even if he did not remember exactly how they were introduced in schools. This is the case of all teachers during the 1<sup>st</sup> interview.

FE-MK2 – The teacher explains another way of organising the first encounter (all teachers, 3<sup>rd</sup> interview).

FE-T1 - The teacher explains the way limits are usually taught in Mozambican secondary schools and has not figured out another way of organising the first encounter (FE-MK1). As a consequence he either taught limits in the same way (Abel, 1<sup>st</sup> interview), or we can surmise that he would organise students’ first encounter with limits of functions in the same way (David, 1<sup>st</sup> interview).

FE-T2 – The teacher explains the way limits are usually taught in Mozambican secondary schools (FE-MK1) and states that it does not help students understand the concept. In this category I classified a teacher who referred to students’



difficulties in general (teacher's point of view) and not only his own difficulties as a student (student's point of view). My indicators are utterances such as: "It's difficult for students to understand" (Abel, 1<sup>st</sup> interview), "What is missing is understanding what the limit concept is" (Frederico, 1<sup>st</sup> interview) "One of the main problems that students face ..." (Mateus, 1<sup>st</sup> interview).

FE-T3 – In order to overcome students' difficulties (FE-T2) the teacher is able to suggest slight changes in the way the first encounter with limits is organised but does not elaborate a new introduction. My indicators are utterances such as: "If we had a graphical representation ..." (Mateus, 1<sup>st</sup> interview), "Reading graphs should be more refined work" (Frederico, 1<sup>st</sup> interview).

FE-T4 – The teacher is able to challenge the usual first encounter with limits and suggest a new way of introducing limits. My indicators are utterances such as: "I will start with the numerical setting (...) then I will use the graphical method" (Mateus, 3<sup>rd</sup> interview), "I would use the numerical setting" (David, 3<sup>rd</sup> interview), "I would prepare a task (...) using intuitive ideas" (Frederico, 3<sup>rd</sup> interview).

FE-T5 – In addition to the explanation of a new way of organising the first encounter with limits (FE-T4), the teacher is able to defend his ideas using strong arguments. My indicators are utterances such as: "I think that the graph has more impact" (Mateus, 3<sup>rd</sup> interview), "I believe that with the numerical setting I could easily explain what the limit is, and explain its link with the term 'approximation'" (David, 3<sup>rd</sup> interview), "We are directly observing the graph (...) the student can easily understand (...) what 'tends to' means" (Frederico, 3<sup>rd</sup> interview).

FE-T6 – Besides defending a new institutional relation to limits (FE-T5), the teacher is able to analyse possible deviations and suggest ways of avoiding them. This is the case of this statement: "The examples used (...) are only functions with a limit (...) as a teacher, I begin with a function which has a limit and another function ..." (Mateus, 3<sup>rd</sup> interview).

I will now explain how, for each teacher, I classified his prior and final knowledge across the data analysis, using these categories.

## 7.2 First interview

### 7.2.1 Abel

Abel had been in contact with limits of functions through three institutions, two as a student and one as a teacher.

#### *Teacher training course at EMU, as a student of the Faculty of Education*

This was Abel's first contact with limits. During the first interview, he tried to remember how this concept was introduced during the course, speaking about function, application, graphs, neighbourhood, calculations, accumulation point, convergence, divergence, but without any clear relation between them.

[suspiros] Ah, acho que foi ... deram-nos ... deixa ver ... uma função ou uma aplicação, Oh meu Deus! ... suponho eu que era assim [desenha um diagrama na folha; Appendix 7.1] o  $x$ , depois aqui  $f(x)$  ... depois vinham uns gráficos. Recordo-me que devia ser, primeiro o método ... aplicação (...) Depois era  $x_n$ , depois a representação gráfica (...) Eu não me recordo se [o professor] teria falado sobre a vizinhança naquela altura, não me recordo (...) Mas uma coisa assim, depois fazíamos os cálculos, tínhamos aqui o ponto de acumulação (...) Agora, se demos a vizinhança, não sei precisar neste momento (I1/A/211-247).

[sighing] Ah, I think that it was ... they gave us ... let me see ... a function or an application, Oh my God! ... I guess it was like that [he draws a diagram on a sheet of paper; Appendix 7.1]  $x$ , then here  $f(x)$  ... then came some graphs. I remember that it might be first the method ... application (...) Then it was  $x_n$ , and then the graphical representation (...) I don't remember whether he [the teacher] spoke about neighbourhood at that time, I don't remember [...] But something like that, then we made calculations, we had the accumulation point here (...) Now, if we spoke about neighbourhood, I can't say for sure any more.

In this quote, Abel does not seem to have a clear idea on how the limit concept was introduced during his first training course at EMU. He seems to be using his memory of keywords introduced in the lessons rather than his understanding of the limit concept. They probably began with graphs of sequences as he stated later, when analysing the definitions, “first we spoke about sequences, and then we went on to ...” (“nós falamos primeiro de sucessões, depois fomos entrando”, I1/A/272-273), and were given the definition by Heine<sup>14</sup>. He remembered that the

<sup>14</sup> Definition by Heine

We say that  $f$  (a real function of a real variable belonging to a domain  $D$ ) has limit  $L$  when  $x$  tends to  $a$  if for each sequence  $x_1, x_2, x_3, \dots, x_n$  (belonging to  $D$ , different from  $a$ ) converging to  $a$ , the sequence  $f(x_n)$  converges to  $L$ . We write  $\lim_{x \rightarrow a} f(x) = L$ .

teacher used very small values of  $x$  and asked students to calculate the corresponding values of  $f(x)$ .

Recordo-me que o professor dava valores muito pequeninos para dar a entender ... As vezes era 0, não sei quantos, 0,001 (...) Valores muito pequeninos para dar a entender, euh, este conceito, esta aproximação (I1/A/350-67).

I remember that the teacher used very small values for us to understand ... Sometimes it was 0, something, 0.001 (...) Very small values for us to understand, er, this concept, this approach.

He said that he made the required calculations without understanding their meaning (I1/A/379-84).

Later on during the interview, Abel recognized the numerical tasks (Appendix 6.3, Sheet 2) as being what they did at the Faculty of Education. He explained that the teacher gave the  $x$ -values for the students to calculate the  $y$ -values (I1/A/850-54), and agreed when I suggested that it would be better if the students were actively involved, choosing the values themselves (I1/A/856-67).

Abel also said that he considered the limit concept as a very difficult one (I1/A/388) and only memorized the techniques (I1/A/620-21).

Abel has an imprecise idea of his first contact with limits of functions. It seems that the teacher tried to give an intuitive idea of limits through numerical calculations. However, it appears that the students did not engage in a discovery process, such as choosing  $x$ -values, calculating the corresponding  $y$ -values and analysing their trend in terms of limit of the function. This could explain why Abel did the calculations but did not understand their meaning.

### ***Upper Pedagogical School in Germany as a student***

Abel said that he had many difficulties because he had to study in German, which was a new language for him. He said that they used the graph, the definitions by Heine<sup>14</sup> and another definition, which he did not remember (I1/A/421-22), but recognised this as the  $\varepsilon$ - $\delta$  definition when I showed him Sheet 1 (I1/A/438). He said that he had difficulties in understanding the  $\varepsilon$ - $\delta$  definition and, as a consequence, turned to memorization.

Bom ... na altura, houve muitas dificuldades, não é, mas prontos, na altura não estava assim tão bem, não é ... mas,

Well ... at that time, it was very difficult, you know, at that time I was not that good, you know ... but, with the notions that I

com as noções que tinha, mais as explicações dos colegas, fui entendendo, fui entendendo. Bom, por outro lado também a língua. A princípio só foi quase que, euh, decorar um pouco (I1/A/450-53).

already had, and my colleagues' explanations, I got to understand, I got to understand. Well, on the other hand the language as well. At the beginning it was like, er, memorizing.

This second contact with the limit concept was less intuitive and more formal, with the definition by Heine and the  $\varepsilon$ - $\delta$  definition. This way of introducing limits does not seem to have changed Abel's conception of limits. The memorization of techniques remained dominant.

### ***Mozambican secondary school, as a teacher***

Abel said that he taught limits in secondary schools in Quelimane (1990-1994) and in Maputo (1995-2002) (I1/A/137-183). His students had difficulties in understanding this concept and he thought that it could be because of his way of teaching.

Noto que, na escola, para introduzir este conceito, tem sido um pouco, pouco difícil, para entenderem, os alunos. Ya. Bom, se isso só é comigo (I1/A/576-77).

I observed that in schools, to introduce this concept, it has been kind of, kind of difficult, for the students, to understand. Yes. Well, whether it is only with me.

He spoke of introducing the limit concept using another method, but did not know which method.

Então, não sei se, talvez, eu ... em vez de utilizar, euh, este método, talvez utilizando um outro método, não sei que método neste momento, talvez a partir, como estava a dizer a partir de sucessões, em vez de só, só a partir, a partir de, de gráfico, talvez a coisa seja um pouco ... fácil (I1/A/585-88).

So I don't know whether, maybe, I ... instead of using, er, this method, maybe using another method, I don't know which method at this stage, maybe starting, as I was saying starting with sequences, instead of only, only starting, starting with, with graphs, maybe it would be a little ... easier.

In this quote, Abel suggests that he introduced limits starting with graphical representations, but considered that starting with sequences could be easier for students. Nevertheless, according to the Mozambican syllabus for secondary schools, limits are introduced through sequences. Therefore I asked him how he introduced limits in schools.

I: Qual era o seu ponto de partida? Era o gráfico, eram as sucessões, era a definição?

I: What was your starting point? Was it the graph, the sequences, the definition?

A: Primeiro comecei com sucessões (I1/A/594-96).

A: First I began with sequences.

It seems that there is some contradiction between this statement and the previous one. Abel now said that he started with sequences, while in the previous quote he presented the introduction through sequences as an alternative to an introduction through graphs. An explanation for this contradiction could be that Abel had not been teaching for several years when I interviewed him for the first time. He did not tell me that until the last interview so, at the time of the first interview, I thought that he was currently teaching in a secondary school in Maputo.

I then asked him when he got the idea of using another method. Did it come from our first meeting or did he have this idea before? He answered that he had this idea before, and explained.

Porque, eu via a coisa, explico, mas ... parece que não fica assim bem entendido. Mas também começo a reflectir, quando era estudante, bom aquilo foi, encaixei e prontos (I1/A/620-21).

Because I saw it, I explain, but ... it seems that they don't understand properly. But also when I think about it, when I was a student, well it was, I put it into my head and that's it.

I then asked him if it was for this reason that he chose this topic for his research. Was it to meet a worry that he already had?

Sim. Agora se vou conseguir essas outras alternativas (...) Não sei! (I1/A/637-41)

Yes. Now if I will get these other alternatives (...) I don't know!

As we spoke quite a lot about the first encounter during the first part of the interview, we did not come back to this point during the second part.

### ***Overview***

In conclusion, at the beginning of the research, Abel knew the way limits are usually taught in Mozambican secondary schools, and was aware that this way of introducing limits does not help students understand this concept (it has been “difficult for the students to understand”). This idea came on the one hand from his experience as a student in Mozambique and Germany, where he memorized definitions and techniques instead of understanding the concept. On the other hand, as a teacher he observed that his students faced the same problem. It could be one of the reasons why he entered the group and chose the topic “Alternative ways of introducing the limit concept”.

Nevertheless, he did not seem to know an alternative way of introducing the limit concept other than through sequences, as stipulated in the Mozambican syllabus

for secondary schools. Furthermore, Abel appeared to be a rather unconfident person. As a teacher, he was wondering whether students' difficulties with the limit concept only occurred with him ("whether it is only with me"). As a researcher, he was not sure whether he would find other ways of introducing this concept ("Now if I will get these other alternatives [...] I don't know").

For these reasons, I classified his prior mathematical knowledge as FE-MK1 and teaching ideas as FE-T2 (see pages 189-190).

### 7.2.2 Mateus

Mateus had been in contact with the limit concept through three institutions, always as a student. He had never taught limits.

#### *Mozambican secondary school as a student*

He did not remember how this topic was introduced during this first contact with limits. He only remembered doing calculations.

Acho que fomos directamente na, nas, nos modelos, eh, algébricos (I1/M/228).	I think that we went straight to, to, to, er, to algebraic models.
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Mateus used the word "model", probably as a reference to his own topic "different settings and models".

#### *Nautical School as a student*

At the Nautical School, they also worked in an algebraic setting.

Na Escola Náutica também foram modelos algébricos (I1/M/232-3).	At the Nautical School too it was algebraic models.
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Era mais, são mais modelos algébricos, calcula, calcula (I1/M/247-8).	It was more algebraic models, calculate, calculate.
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He studied the  $\varepsilon$ - $\delta$  definition, mainly during the second year (I1/M/304-8) but did not understand it then, nor now (I1/M/342-3).

#### *Teacher training course at PU as a student*

At the Pedagogical University, they also "went straight to the algebraic model" (I1/M/270-75). Mateus remembered having a "good lecturer" ("um bom professor", I1/M/351), who gave them tasks to calculate  $\delta$ , but he did not understand these tasks. They never used the results of a limit.

(...) mas ali na minha casa [Universidade Pedagógica] é mais algébrico [...] não, é só calcula, comparar nas soluções do Demidovitch e está certo (420-28).

(...) but here in my house [meaning the Pedagogical University] it is more algebraic [...] it is only calculating, comparing the result with Demidovitch, and that's it.

As I showed in Chapter 2, the textbook by Demidovitch strongly influenced the Pedagogical University's institutional relation to the limit concept.

### *Second part of the interview*

Speaking about students' difficulties at the end of the interview, Mateus criticized the way limits are usually introduced in Mozambican secondary schools as both algebraic calculations and the  $\varepsilon$ - $\delta$  definition.

Um dos maiores problemas que nós, euh, alunos temos ... a partir do, do conceito do próprio limite ... porque ... muitos dos nossos professores começam só pelo cálculo algébrico ... É verdade que nós, pode aparecer essa definição  $\delta$ - $\varepsilon$ , mas interpretar isso o que é! (I1/M/1616-28)

One of the main problems that we, er, students face ... starting with, with the limit concept itself ... because ... many of our teachers only begin with algebraic calculations ... It's true that we, we can learn this  $\delta$ - $\varepsilon$  definition, but to interpret its meaning, that's another story!

In this quote, Mateus was giving an opinion about the introduction of limits in Mozambican secondary schools, using his own experience as a student: it does not lead to any interpretation of limits. He was able to suggest another kind of approach, using several registers such as the numerical and the graphical registers.

Eu acho que se comessem pelos gráficos talvez ... porque, como se começa na, eu aprendi na 10<sup>a</sup> classe nas sucessões ... sucessões, normalmente começa-se por longa tabela (...) então a partir dali vê-se o comportamento dos valores, então a relação entre os valores (...) e depois também, se houver depois uma construção gráfica ... então para ver que, quando uma tabela depois um gráfico mais ou menos mostrar que depois esses valores estariam assim (...) porque depois o outro problema é que euh, não, que eu tenho como aluno, a dificuldade é que não se pega outros modelos senão só o método algébrico (I1/M/1628-65).

I think that if they began with graphs maybe ... because, starting with, I learnt in Grade 10 with sequences ... sequences, usually it starts with a huge table (...) then we can see the behaviour of the values, then the relation between the values (...) and then also, if we had a graphical representation ... to see that, when having a table and then a graph showing more or less that these values would be like that (...) because another problem is that, er, no, that I have as a student, the difficulty is that we don't use other models, only the algebraic model.

In this quotation, Mateus said that his study of limits began with sequences, while in previous quotes he spoke about the  $\varepsilon$ - $\delta$  definition and algebraic calculations. There can be two explanations for this apparent contradiction. One the one hand,

as already explained, the teachers were recalling their learning of limits during the interview, without previous notice. Mateus had possibly forgotten some aspects of this first encounter with limits and remembered it later. Another explanation is that he did not consider sequences as functions. In the Mozambican syllabus, there is a dichotomy between sequences and functions (meaning functions of a real variable). For this reason many students, and maybe some teachers, do not consider sequences as functions. In that case he would be suggesting here that functions could be introduced in the same way as sequences, using numerical and graphical approaches.

As these statements were made at the end of the interview, we can surmise that Mateus was influenced by the tasks that I presented to him during the interview. Even so, he was able to use this new information to reorganise his prior knowledge. He also suggested a more geometric approach but did not explain what “geometric” meant for him. It seems that he was also speaking about graphs.

Talvez se na própria introdução do conceito limite fosse mais por esse método ... de tabelas e, talvez também método geométrico (...) ver mais ou menos a partir do gráfico o que é que nós chamamos de limite (I1/M/1722-1728).

Maybe if the introduction of the concept used more this method ... of tables and, maybe also the geometric method (...) to see more or less from the graph what we call limit.

### **Overview**

To sum up, Mateus remembered learning limits through algebraic calculations (at school and at university level) and the  $\varepsilon$ - $\delta$  definition that he did not understand (at the Nautical School and at university). I classified his mathematical knowledge at the beginning of the interview as FE-MK1 (see page 189).

At the end of the first interview, and after analysing the tasks that I presented to him, he also recalled using the numerical and graphical register when studying limits of sequences. He was able to criticize the traditional way of introducing limits in Mozambican institutions, and to suggest other methods, using several different registers. It is not easy to know whether Mateus had these ideas before or whether he began thinking about these issues during the interview. His personal relation to the limit concept had possibly evolved through our discussion and the tasks I presented to him.



Although Mateus did not have Abel's teaching experience of limits, he seemed to be able to reflect more in depth about the way limits are approached in Mozambican secondary schools, using his own experience and the tasks presented during the interview. He also appeared to be more confident, suggesting new ways of introducing the concept. I classified Mateus' ideas about how to organise the first encounter as FE-T3 (see page 190).

### 7.2.3 David

David was the youngest of the group. He had only met limits in two institutions as a student: secondary school and the Pedagogical University. He had never taught limits.

#### *Mozambican secondary school as a student*

David remembered that at secondary school, they carried out calculations using procedures taught by the teacher.

Bom, euh, a ideia que fiquei na altura é que prontos, não é, é que os procedimentos já estavam construídos, eram-nos dados como ferramentas que nós tínhamos que usar para um determinado objectivo (I1/D/63-65).

Well, er, the idea that I got at that time is that, ok, you know, is that the procedures had already been constructed, they were given to us as tools that we had to use for some objective.

This quote is a good illustration of the way mathematics is taught in Mozambique, and in many other countries. A specific procedure is associated with a specific kind of task, as if it always existed. Students are not expected to find any solution by themselves, but only to apply the taught procedure. It is what Chevallard (1999) called the *cultural-mimetic problematic* (see page 73).

David also said that he did not understand the objective of these calculations, because they never used the results.

... ficava sempre aquela dúvida: porque é que estou a usar isto aqui, para que é que estou a calcular o limite duma função ... ou o limite duma sucessão, porque é que estou a calcular? (I1/D/65-74)

... I always was uncertain about the reason why I was using that, why was I calculating some limit of function ... or some limit of sequence, why am I calculating this limit?

This utterance illustrates an important consequence of the institutional relation to limits, and many other concepts. The calculations are considered as the aim of the mathematical work, instead of a means to get some result. As a consequence,

knowledge is compartmentalised, and students are not able to connect and/or use what they have learnt.

In addition to calculations, they were requested to apply the definition.

Cálculos e, definição, aplicar a definição do próprio limite (...) mas no fundo, não, não, não nunca vim ganhar uma ideia muito fixa de porque é que estou a calcular este limite (I1/D/91-97).

Calculations and, definition, to apply the definition of the limit (...) but in the end, I didn't, I didn't, I never got any clear idea of why I was calculating those limits.

I then asked him whether they used the result of a calculation, for example to sketch a graph. He answered that they used some graphs at the beginning to illustrate limits but did not use limits for sketching graphs.

A princípio quando se estava a construir a própria, o próprio limite, não é, utilizamos o gráfico ... só para mais ou menos ver o que é que acontece a um certo tempo ... Mas prontos, para explicar realmente o objectivo de estarmos a usar o próprio limite nós não chegamos a fazer (...) Era simplesmente cálculos e terminamos por aí (I1/D/102-125).

At the beginning, when we were constructing the limit, the limit itself, you know, we used the graph ... only to see more or less what was going on in some situation ... But then, we didn't get to really explain the objective of using the limit (...) It was only calculations and we stopped there.

He did not remember which definition was given at that time but, looking at the definitions in Sheet 1, he recognised the  $\varepsilon$ - $\delta$  definition as the one he learnt at school. They had to memorise the definition, which was difficult because he did not understand it.

(...) sinceramente falando não percebia mas, é, pronto, é aquilo que estava a dizer, era obrigado a aprender, tinha que fixar numa ou noutra maneira (I1/D/168-246).

(...) sincerely speaking I didn't understand but, it is, I mean, it is how I said, I had to learn it, in some way I had to fix it in my memory.

### ***Teacher training course at PU as a student***

David said that the teaching of limits at PU was not very different from at secondary school.

Já na Universidade Pedagógica, a história não, não se diferenciou tanto do que nós já havíamos vistos no ensino secundário porque, é aquilo que já se diz, que isto aqui é uma revisão dos conteúdos já tratados a nível secundário (I1/D/277-79).

Now at the Pedagogical University, the story was, was not different from what we saw at secondary school level because, that's what they say, that this is revisiting the contents given at secondary school level.

He said that they also studied the  $\varepsilon$ - $\delta$  definition. I asked him whether he understood it better and he answered “I still don’t understand it” [Continuo sem perceber] (I1/D/330).

### *Second part of the interview*

During the second part of the interview, I asked David his opinion about the first encounter with the limit concept in Mozambican schools.

I: (...) O que é que acha dessa abordagem. Acha boa, acha que ...

D: Prontos, euh, a princípio é um bocado difícil estar aqui a dizer o que é que eu acho porque, é aquilo que estava a dizer, é um dado já adquirido e, já é sistemático, os professores sempre usam! Por acaso estava a rever um caderno do meu irmão mais novo, não é, que ele está a fazer o 12º ano

I: Ah, é interessante!

D: Então a sequência é sempre a mesma

I: É sempre a mesma, não é!

D: Então, prontos. É a única base que se utiliza por isso que as coisas agora começam a limitar-se ... Se existirá outra maneira de poder se introduzir ou puder-se trabalhar os limites fora desta sequência, porque prontos, dá-se um certo, certos valores, depois começa a ver mais ou menos como é que é a sequência dos próprios valores (I1/D/410-445).

I: (...) What do you think of this approach? Do you think it is good, do you think that

...

D: Well, er, it’s a little bit difficult to say here what I think because, it’s what I was saying, it’s a given fact and, now it is always like that, teachers always do that! By chance I was looking at my younger brother’s exercise book, you now, who is in Grade 12

I: Ah, interesting!

D: Then the sequence is always the same

I: It’s always the same, isn’t it!

D: Then, well. It is the only base used because now things are limited ... Whether there is another way for introducing or for working with limits out of this sequence, because well, some, some values are given, then we began to see more or less what the sequence of the values themselves is.

This quotation shows that David knows the way limits are usually introduced in secondary schools. He even looked at this part in his brother’s exercise book and recognized the way it has been introduced to him at school. This way of introducing limits in Mozambican schools seems to be immutable. It is a strong institutional relation. Moreover, David did not refer to students’ difficulties related to this introduction.

### *Overview*

To sum up, David’s first contacts with limits through two institutions was mainly algebraic and formal. He remembered having some graphical interpretation done at school, but this was not followed by any further graphical task. Up to this point he personally did not understand the  $\varepsilon$ - $\delta$  definition, or the meaning of the

calculations, but he did not speak about students difficulties. He seemed to consider the usual approach of limits as immutable and he did not present any alternative to introduce limits of functions in schools. Unlike Mateus, his knowledge about the first encounter did not appear to evolve during the interview. I therefore classified his mathematical knowledge as FE-MK1 and his ideas about teaching as FE-T1 (see pages 189-190).

#### 7.2.4 Frederico

Frederico met the limit concept through two institutions as a student and one institution as a teacher.

##### *Teacher training course at EMU, as a student of the Faculty of Education*

Frederico studied limits for the first time during his teaching training course at EMU. It was a long time ago and he was not able to remember how the teacher introduced this topic. He remembered not having a very “concise” [probably meaning precise] idea about limits. He remembered studying the  $\varepsilon$ - $\delta$  definition, but was not sure about using it.

Bem acho que a definição, demos a definição ... acho eu, uma definição talvez matemática só, euh ... não me recordo se nessa altura, euh, chegamos a fazer algumas demonstrações sobre ... euh, a definição ... Como disse, euh, talvez se eu tivesse, euh, aberto os cadernos (I1/F/225-228).

Well, I think that the definition, we gave the definition ... I think, maybe a mathematical definition, er ... I don't remember whether at that time, er, we made some proofs with ... er, the definition ... As I told you before, er, maybe if I had, er, opened my exercise books.

Frederico did not seem to have a precise idea of this first encounter with the limit concept.

##### *Agricultural School as a teacher*

At the Agricultural School Frederico taught limits for one year, even if it was not part of the syllabus. He started with numerical sequences.

Bem como se [rindo] tratava-se de ... de, de conceitos elementar, acho que o ... eu dei limite partindo portanto de, de sucessões assim, de ... de uma progressão qualquer ai ... u, usando as, as sucessões de, numéricas ... mais ou menos um, conceitos elementares (I1/F/344-46).

Well as [laughing] it was about ... about basic concepts, I think that the ... I taught limits starting with, with, sequences, with ... with some progression ... using the, the numerical sequences ... more or less, basic concepts.

It was a good experience for him because, when he met these students some years later, they expressed their gratitude for teaching them limits. It had been useful in their further studies.

... quando vieram me encontrar, eles ficaram satisfeitos, que afinal de conta nós pensávamos que o professor estava nós maltratar, quando deu-nos os limites, quando deu-nos aquelas progressões aritméticas e, e geométricas, mas quando nós fomos lá na escola, euh, no, no Instituto Agrário, nós fomos bons à Matemática, e nós gostamos daquilo que o professor fez (I1/F/426-54).

... when they met me later, they were satisfied, that “after all we thought that you were being hard on us, when you gave us all those arithmetic and geometric progressions, but when we got to school, er, at, at the Agricultural Institute, we were good at maths and we liked what you did”.

He said that, except for this numerical work, they did some algebraic calculations, but did not use any graph. They only worked with sequences and did not use other functions.

### ***Teacher training course at PU as a student***

At PU Frederico did not like this topic because he did not understand it. He was not able to explain how the teacher introduced limits but remembered his feelings about it.

... a gente não chegava a perceber o que, o que é que significa, que limite a esquerda, limite a, a, a direita, e como [o professor] chegava e começava a fazer as, as demonstrações, só escrever no quadro aí, demonstrações longas que só ele é que estava a falar, não entendíamos o que, o que é que significava  $\varepsilon$  (...) só víamos as fórmulas e (...) preferíamos que acabasse aquele, aquele capítulo de [ri-se] dos limites, aproveitávamos para fazer outra coisa porque não estávamos a, a entender (...) euh, mais ou menos comecei a entender um pouco quando já estávamos a falar sobre ... as regras usando Cauchy e, euh, d'Alembert (I1/F/541-59).

... we did not understand what, the meaning of, limit to the left, limit to, to, to the right, and how he [the teacher] came and began writing proofs, writing on the board there, long proofs where he was the only one speaking, we didn't understand what, what  $\varepsilon$  meant (...) we only saw formulas and (...) we just wanted to finish this, this chapter on [laughing] on limits, we did anything else during the classes because we didn't understand [...] er, more or less, I began understanding a little when we were speaking about ... the rules using Cauchy and, er, d'Alembert.

In this quote, Frederico is speaking about limits of functions, in particular one-sided limits that he did not understand. He is also confusing limits of a function with the rules to determinate whether series are convergent or divergent (Cauchy and d'Alembert's rules). That was when he came to understand limits, maybe because he could see its usefulness. It seems that by “understanding” Frederico

means understanding how to use the rules (algorithmic understanding) and not understanding the limit concept (conceptual understanding).

The teacher also gave proofs using the  $\varepsilon$ - $\delta$  definition during lectures where students were only required to listen and probably take notes.

É por isso que então nós ficávamos sem saber o que é que significa o  $\varepsilon$ , por exemplo, esse  $\delta$ , o, o quê, só vi, só víamos o professor a fazer demonstrações e chegava ao fim (I1/F/605-14).

It is why we did not understand what this  $\varepsilon$  meant for example, or this  $\delta$ , what, what, we only saw the teacher giving proofs and reaching the end.

The teacher began with limits of sequences, giving proofs (I1/F/575-76).

Frederico also said that they studied the  $\varepsilon$ - $\delta$  definition and did algebraic calculations but without interpreting the answer (I1/F/551).

### ***Second part of the interview***

Speaking about teaching limits in schools during the second part of the interview, Frederico stated that it would be beneficial to work more graphically, because visualising could help students understand the limit concept better.

A leitura dos gráficos, acho que devia se fazer um, um trabalho mais, mais apurado, mas de todas, de todas as formas, euh, acho que, euh, são exercícios que podem ajudar (I1/F/1064-65).

Reading graphs, I think that we should do a, a more, more refined work, but in any, in any case, er, I think that, er, these are tasks that can help.

Acho que, acho que sim, acho que havia de, de, de, de, de ajudar muito porque eu o, o, o que falta é ... entender o que é que significa a, a, esse conceito de limite. Porque assim com cálculos só, eles não chegam a entender. Pelo menos aqui já nos gráficos visualizados, consegue ver o que é que acontece de facto quando os valores de  $x$  estão a crescer, o que é que acontece com, com, com, com os valores de, da, por exemplo aqui eles podem ver que, quando  $x$  tende para esse valor  $a$  (I1/F/1235-40).

I think that, that, yes, I think that it could help a, a, a, a, a lot because I, what, what is missing is ... understanding what the, the, this limit concept means. Because doing only calculations, they [students] do not understand. At least here, with visualised graphs, they can see what happens in fact when the  $x$ -values are increasing, what happens with, with, with, the values of, of, for instance here they can see that when  $x$  goes to this value  $a$ .

As with Mateus, Frederico could have been influenced by the graphical tasks that I showed him before he made these statements. He did not speak about other ways of introducing this concept.

### ***Overview***

To sum up, at the beginning of the research Frederico did not recall exactly how limits were introduced at the Faculty of Education and at PU. He remembered doing calculations and studying the  $\varepsilon$ - $\delta$  definition that he did not understand. As a teacher at the Agricultural School, he successfully introduced limits through sequences, out of the school's syllabus, but using the way limits of sequences are usually taught in secondary schools. At the end of the interview, he said that visualising limits through graphs could help students understand this concept, maybe influenced by the tasks he was shown during the interview. As in Mateus' case, I therefore classified Frederico's mathematical knowledge as FE-MK1 and his ideas about teaching as FE-T3 (see pages 189-190).

### **7.2.5 Summary**

At the beginning of the research, all four teachers remembered having learnt limits at school through calculations, which were never used in practice, and the  $\varepsilon$ - $\delta$  definition, which they did not understand. This is consistent with the study of Mozambican didactic institutions' relation to the limit concept as described in Chapter 2. Some of them also mentioned the two textbooks that I analysed in that chapter as being used by their teachers: the textbook by Demidovitch (I1/M/407, 428; I1/D/384; I1/F/668, 672, 676, 688) and the textbook by Piscounov (I1/M/295-96; I1/F/689).

Abel experienced another introduction to this concept at the Faculty of Education, using numerical values, but did not understand it. As a consequence, he turned to memorization.

Two of the teachers taught limits in schools. Abel, as a Grade 12 teacher, taught the  $\varepsilon$ - $\delta$  definition and algebraic rules to calculate limits, and was aware that his students had difficulties in understanding this concept. It may be for this reason that he chose the topic "Alternatives ways of introducing the limit concept" for his dissertation.

Unlike Abel, Frederico had a good experience of teaching limits at the Agricultural School, out of the syllabus. His students were grateful to him for teaching them this topic.

It is interesting to compare these two teaching experiences. As a Grade 12 teacher, Abel had to teach according to a syllabus and institutional constraints, such as time restrictions and having to prepare his students for the final examination. As I showed in Chapter 2, in the final examinations students are required to calculate indeterminate forms using specific techniques. Abel was aware that his students did not grasp the meaning of the concept, but did not have much room, as well as personal knowledge, to use other ways of introducing the concept. He felt that limits should be introduced in a different way, but did not picture which method could be used. Frederico was in a different position. He taught limits out of the syllabus to help his students in further studies. He did not have many institutional constraints and provided an intuitive introduction to the concept, using numerical calculations with arithmetic and geometric progressions. These two experiences clearly show how the institutional constraints weigh on teachers' choices.

All four teachers seem to be critical about the way limits are taught in schools, at different levels. Three of them (Abel, Mateus and Frederico) challenge this institutional relation. Abel does not picture other ways of organising his students' first encounter with limits, but Mateus and Frederico were able to suggest the use of graphs at the end of the interview. They probably had been influenced by the graphical tasks presented during the interview and, for Mateus, by his own research topic: "Different settings and models to study limits of function in schools". David, as a student, was aware that he did not understand the concept. Nevertheless he did not challenge the institutional relation.

At the beginning of the research process, none of the teachers seemed to know of ways of introducing limits in schools other than the one usually taught in Mozambican secondary schools. However some of them were able to challenge the institutional relation, even if they were not able to articulate suggested changes. The classification of the four teachers' personal relation to the first encounter with limits, using my categories is presented in Table 7.3.



**Table 7-3 Teachers' prior personal relation to the first encounter with limits  
– First interview**

	Mathematical knowledge	Teaching ideas
<b>Abel</b>	FE-MK1	FE-T2
<b>Mateus</b>	FE-MK1	FE-T3
<b>David</b>	FE-MK1	FE-T1
<b>Frederico</b>	FE-MK1	FE-T3

### 7.3 Abel's second interview and dissertation

As it was the topic of his research, Abel obviously had to look for and reflect on different ways of introducing the limit concept in secondary schools. In his dissertation he presented different possible introductions of this concept. They were:

- (i) Introduction through the tangent line (geometrical setting);
- (ii) Introduction through a graph (graphical setting);
- (iii) Introduction through a rational function (numerical setting);
- (iv) Introduction through instantaneous velocity (kinematics setting);
- (v) Introduction through a combination of settings (numerical, graphical and formal settings).

In previous versions of his work, he presented two other alternatives: introduction of the limit concept through the  $\varepsilon$ - $\delta$  definition and introduction through a sequence (numerical and graphical settings).

The use of the  $\varepsilon$ - $\delta$  definition as an introduction to limits was abandoned in October 2004. In fact it was not an introduction of the limit concept through the  $\varepsilon$ - $\delta$  definition, but an introduction of the  $\varepsilon$ - $\delta$  definition. In the chapter as discussed during the 12<sup>th</sup> seminar, it reads:

A definição formal usa uma linguagem simbólica  $\varepsilon/\delta$  difícil de ser interpretada tornando a introdução do conceito de limite também difícil de ser compreendido. Entretanto, a definição formal não pode ser

The formal definition uses a  $\varepsilon/\delta$  symbolic language that is difficult to interpret, which makes the limit concept difficult to understand. However the formal definition cannot be eliminated because it is the

eliminada pois é linguagem que os matemáticos utilizam para definir formalmente limites.

language that mathematicians use to formally define limits.

As alternativas anteriormente descritas poderão ser usadas visando chegar à definição formal duma maneira mais compreensível. Recomenda-se que esta alternativa seja ensinada depois de os alunos terem compreendido as alternativas anteriores.

The alternatives previously described could be used in order to reach the formal definition in a more understandable way. I suggest that this alternative be taught after the students already understood the previous alternatives.

The first part of this quotation illustrates the conviction shared by many teachers in Mozambique that the  $\varepsilon$ - $\delta$  definition must be taught at secondary school level.

This conviction comes on the one hand from the syllabus, which stipulates (explicitly until 1993, and then implicitly) that the  $\varepsilon$ - $\delta$  definition should be taught (see pages 33-34). To be a “good subject” of the institution, a teacher should teach it. On the other hand, as stated by Abel, “it is the language of mathematicians”. Not to teach the  $\varepsilon$ - $\delta$  definition would locate the knowledge taught in secondary schools too far from the reference that legitimates this knowledge.

The second part of this quotation illustrates the contradiction already mentioned in Chapter 2 (see pages 18 and 33) between two constraints of the didactical transposition: the knowledge must be justified; hence the teaching of the  $\varepsilon$ - $\delta$  definition; but it must also be understandable. Abel concludes here that the  $\varepsilon$ - $\delta$  definition must be taught after a more intuitive introduction of the limit concept. It is not suitable to introduce the concept through the  $\varepsilon$ - $\delta$  definition at secondary school level. For this reason, this alternative was removed in further versions of his work.

At the time of the second interview, Abel had already completed a literature review of several ways of organising the first encounter with limits, interviewed some secondary school teachers about the way they usually introduce the limit concept, attended some classes about limits in a private secondary school in Maputo, and was preparing his experiment. This work helped him reflect on his prior knowledge, as shown in the following quotation.

Só agora quando comecei a ver alguns livros aí na biblioteca, euh, os livros de Larson, outros de, que falam sobre o conceito de, de limite geralmente vem

Only now when I started looking at some books in the library, er, books by Larson, others, which speak about the limit concept, usually found in “Calculus”, it is

"Calculus", só aí consegui ver que afinal há outras alternativas. Porque antes eu só sabia que bastava explicar através daquele gráfico, e dar a definição, prontos, começava com os exercícios e acabou. Mas que houvesse por exemplo o método, euh, euh, numérico que poderia aplicar, o método, euh, o quadro numérico, o geométrico e, o gráfico prontos é esse que sempre aplicamos, não é, essas outras, essas outras alternativas eu, portanto, não conhecia (I2/A/74-87).

when I saw that there are other alternatives. Because before I only knew that it was enough to explain through that graph, and give the definition, ok, I began with the tasks and that's it. But that there was, for example, the method, er, er, numerical method, er, the numerical setting, the geometrical one and, the graphical one, ok it is the one that we always use, you know, these others, these other alternatives I, I mean, I did not know them.

This quote clearly confirms that prior to the research, Abel only knew the institutional way of organising the first encounter with limits. He discovered new ways of approaching the concept through his literature review, but also through his contacts with a teacher in a private school in Maputo<sup>15</sup>.

Há uma professora que fez um trabalho em relação a, a, a esse conceito, formas de introdução e que já está a aplicar essa, essa metodologia (...) Eles começam mesmo desde o princípio a partir de funções lineares e por aí em diante! ... Sim. Não, não, não se segue, euh, a, ao método de sucessões e depois a partir daí entra-se na, na função, no limite de uma função, não. Eles dão sucessões, sim senhor, mas depois voltam de novo a falar de funções a partir, limite de função a partir da função linear até outras funções. Então introduzem desde o princípio. Então, para dizer que, de facto, eu acho que é um bom método, sem dúvidas, porque o aluno desde, desde o princípio que vai, ao longo do tempo trabalhando sempre com este conceito de limite até as funções mais complexas (I2/A/89-119).

There is a teacher who did some work about, about this concept, ways of introducing, and who is applying this, this methodology (...) They start from the beginning from linear functions and go on! ... Yes. They don't, don't follow, er, the method of sequences and then from there enter the, the function, limit of a function, no. They give sequences, yes they do, but then they come back to the functions using limit of a function going from linear functions to other functions. Then they introduce the concept. Then, this is to say that, in fact, I think that it is a good method, without any doubt, because the student, from, from the start and carrying on over time, working with this limit concept up to more complex functions.

This quotation shows that Abel was discovering a new way of introducing the limit concept in a Mozambican school, different from the way indicated by the syllabus. In fact, according to the Grade 12 Mozambican syllabus, Unit II is dedicated to the study of some real functions of a real variable, before Unit III (Sequences) and Unit IV (Limits and continuity of functions). These functions

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<sup>15</sup> This teacher was my colleague at EMU. We did some research on limits of functions together (Mutemba B. & Huillet D., 1999; Huillet D. & Mutemba B., 2000) and she wrote a Master Degree dissertation about students' understanding of limits (Mutemba B., 2001). For these reasons she teaches limits in a different way.

are: linear function, quadratic function, rational function such as  $f(x) = \frac{ax+b}{cx+d}$ , exponential and logarithmic functions, trigonometric functions and functions with modulus. This study is done without using the limit and derivative concepts. In that school, students were led to analyse the behaviour of Unit II functions in terms of trend, before a formal introduction of the limit concept. The introduction of the concept was then grounded on this experience. As a “good subject” of the institution, Abel had never questioned before the sequence indicated in the syllabus, as he stated.

Eu só sabia que bastava explicar o, euh, a noção de limite de uma função através do gráfico, euh, dava-se a definição e depois entrava-se nos cálculos, isso é o que, bom, era, hum, digamos, que nem, a norma, que eu tinha (I2/A/127-130).

I only knew that it was enough to explain the, er, the notion of limit of a function through the graph, er, the definition was given, and then we started with calculations, this is what, well, it was, er, let say that, like a norm, that I had.

He was amazed because the teacher always asked his students “What does limit mean?” [“O que é isso de limite?” (I2/A/875-76)], and it seemed that students understood (I2/A/870-71). And he concluded:

Eu por exemplo, euh, na altura eu nem me preocupava em exigir ao aluno “diz-me lá o que é isso de limite”! (...) eu achava que não era necessário! Bastava só um aluno calcular-me os limites, prontos, para mi era satisfatório (I2/A/932-42).

In my case for example, er, I even did not worry asking students “tell me what limit means”! (...) I thought it was not necessary! It was enough for him to calculate the limits for me, ok, for me it was satisfactory.


In this quote Abel expresses very clearly what students are usually expected to learn about limits in Mozambican secondary schools: algebraic calculations. Understanding the limit concept is not part of the practice.

In the last version of his dissertation Abel had to reduce the number of alternatives because this part was considered too long by the Faculty of Education Scientific Committee. He chose to remove the introduction through a sequence because, as the introduction through a rational function, it used the numerical register, and could be considered as a variant of this method.

Abel found all these ways of introducing limits in textbooks (Ávila, 1998; Hoffman & Bradley, 1996; Iezzi, Murakami, Hazan & Pompeu, 1985; Morettin, Hazzan, & Bussab 2003; Protter & Morrey, 1977; Silverman, 1969; Swokowski,

1983) and described them in his dissertation. He faced difficulties in analysing these methods in terms of students' difficulties and understanding of the concept. For example, for the introduction through a rational function (numerical register), the following table is presented to determine the limit when  $x$  goes to 1 of the

function defined by  $f(x) = \frac{x^2 + x - 2}{x - 1}$ .

$x$	0.8	0.9	0.95	0.99	0.999	1	1.001	1.01	1.05	1.1
$f(x)$	2.8	2.9	2.95	2.99	2.999		3.001	3.01	3.05	3.10

After concluding that  $\lim_{x \rightarrow 1} f(x) = 3$ , the function is represented as a straight line

with a hole in the point (1 ; 3). The notion of limit is then generalized (p. 24, from Hoffman & Bradley, 1996: 65):

Se  $f(x)$  se aproxima de um número  $L$ , quando  $x$  tende para  $c$  de ambos os lados, então  $L$  é o limite de  $f(x)$  quando  $x$  tende para  $c$ . Este comportamento expressa-se escrevendo  $\lim_{x \rightarrow c} f(x) = L$ .<sup>16</sup>

If  $f(x)$  gets closer and closer to a number  $L$  as  $x$  gets closer and closer to  $c$  from either side, then  $L$  is the limit of  $f(x)$  as  $x$  approaches  $c$ . This behavior is expressed by writing  $\lim_{x \rightarrow c} f(x) = L$ .

Then Abel wrote the following utterance about this method:

A representação gráfica ajuda a visualizar a posição do ponto onde se pretende determinar o limite, por um lado, e os valores na tabela permitem analisar o comportamento da função na vizinhança desse ponto, por outro lado.

On the one hand the graphical representation helps visualize the point where we want to determine the limit and, on the other hand, the table of values allows us to analyse the behaviour of the function in the neighbourhood of this point.

In this quotation, Abel emphasizes the importance of the graph. However, in the method he describes, the numerical representation is dominant, and the graph appears as an interpretation of the numerical work. Whereas, one would have expected him to analyse the role of the numerical register. Furthermore, as Abel said during the first interview, his first encounter with the limit concept at the Faculty of Education had been done through numerical work, but he did not understand the meaning of the calculations because the teacher gave the  $x$ -values and the students only did the calculations (see page 193). Using his own experience, he could have discussed here how to use this numerical introduction in order to give meaning to calculations, for example asking students to choose

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<sup>16</sup> In order to preserve teachers' identity, I cannot provide the reference of their dissertation

the  $x$ -values, and giving more numerical tasks afterwards instead of using only a few examples.

For an experiment in a school, Abel chose to introduce limits of functions through the observation of graphs produced by a computer utility, which was none of the alternatives presented in his dissertation. At that time he was strongly convinced that this method would help students understand the concept, as he stated in his dissertation.

O pesquisador acredita que introduzindo o conceito de limite de uma função utilizando o método gráfico e com ajuda do computador poderá ajudar os alunos a compreender com facilidade o conceito (p. 33).

The researcher believes that introducing the limit concept through the graphical method and using computers would help students easily understand this concept.

And also:

O método de introdução (método gráfico) preconizado na aula experimental foi seleccionado de um conjunto de métodos pouco usados nas escolas, mas pensamos que facilitaria a compreensão do conceito de limite pelos alunos. (...) A construção e visualização dos gráficos foram feitas em computador pela rapidez de processamento de informação e perfeição de construção de gráficos (p. 44).

The method of introducing (graphical method) chosen for the experimental class was selected from a set of methods hardly used in schools, but we thought that it would help students understand the limit concept. (...) The graphs were constructed and visualised by computers for fast information processing and perfect construction of graphs.

When planning his experiment, Abel's knowledge about the first encounter had evolved considerably. He had discovered several new ways of introducing the limit concept, and was convinced that through the graphical method and using computers, students would better understand this concept. I then classified his knowledge before the experiment as FE-MK2 and FE-T5 (see pages 189-190).

## 7.4 12<sup>th</sup> Seminar

During the 12<sup>th</sup> seminar we discussed Abel's chapter about different alternatives of introducing the limit concept. At that time he presented all seven alternatives mentioned in the previous section.

First, the teachers discussed some problems of writing and editing, which are not relevant in terms of their knowledge on how to organise students' first encounter with limits. At the end of this discussion, Mateus made the following suggestion:

M: Agora talvez é a questão da, bom, vamos tentar ajudar a ele, sequência da, das alternativas. O que é que tem que ser primeiro, o que é que tem que ser segundo, para o, o bem do próprio aluno, não é (S12/1821-28).

M: Now maybe the question of, well, we must help him, the sequence of, of the alternatives. What should be first, what should be in second place, for, for the student's benefit, you know.

He then tried to indicate in which order these alternatives should be taught in the classroom (S12/1828-40). I explained that they were “alternatives”, not to be used at the same time. We could discuss in which order to present them in Abel's dissertation, but not in which order to teach them (S12/1841-54). Frederico also explained that the issue was to choose the best alternative for students (S12/1857-61)

This episode is interesting because of Mateus' idea of using all the alternatives with the same students. The same phenomenon occurred during Abel's 3<sup>rd</sup> interview, as I will relate later in this chapter. This leads me to some comments. On the one hand, there is a linguistic problem. In Portuguese, the word “alternative” indicates a choice, and not a simultaneous or sequential use of the different alternatives. It seems that this is not clear for some teachers. This can have strong consequences for mathematics teaching, where we often have to distinguish several alternatives which exclude each other. On the other hand, there is a practical problem. It seems impossible to use all these alternatives with the same class. It would be time consuming and probably confusing for students. Abel, who already taught limits in schools, will actually worry about that during the third interview. Mateus never taught limits and did not seem to realize this fact.

Then I asked the question: “If you should introduce the limit concept in schools, which of these alternatives would you like to try using?”

David stated that he would start with the numerical register. He argued:

D: Isso porque acho que o quadro numérico é mais, quer dizer, do meu ponto de vista, não é, é fácil ver qual é que é a tendência mesmo dos valores dos diferentes, porque ... esta da recta tangente ... sabemos que, em termos de Geometria ... os nossos alunos pouco ... percebem. (...) eu preferia mesmo para o quadro numérico, talvez pela

D: This is because I think that the numerical setting is more, I mean, from my point of view, it's easy to see the trend of the values, because ... the tangent line ... we know that in terms of geometry ... our students don't understand too much (...) I would prefer the numerical setting, maybe because it is easy for students to calculate

faculdade de calcular valores que os alunos têm ... porque de facto o quadro, o quadro gráfico, também seria de prever que ... gráfico é gráfico (S12/1868-81).

values ... because in fact the graphical setting, we have to be aware that ... graph is graph.

David is aware of students' difficulties in geometry and with graphic representations, and for this reason he would not use the introduction through graphs or through the tangent line problem. In fact David himself faced many difficulties with graphs when working on his dissertation, as we will see in Chapter 10.

Then Frederico argued that he would choose to start with graphs. He explained:

F: Ajudava muito mesmo a observar e interpretar aquilo que está a acontecer. (...) Numérico porque, euh, para mi, eu acho que é só calcular, calcular, mas não, euh, ver o que é que está a acontecer no concreto, ao passo que o método gráfico já, pelo menos é um facto! (S12/1892-95)

F: It would help a lot to observe and interpret what is going on. [...] Numerical because, er, for me, I think that it is only calculating, but not, er, to actually see what is going on, while the graphical method, at least it is a fact!

Frederico had already expressed his idea that working with graphs could be helpful for students during the first interview. It seems that he reinforced this opinion during the research process.

A discussion took place between David and Frederico, David standing up for numerical methods and Frederico for the graphical ones (S12/1896-1919).

David's argument was that students face many difficulties in constructing graphs. Frederico argued that when working with numbers the student does not "see" the limit. S/he does not reach the meaning of the limit concept. David answered that he would begin with numbers but link with the graph afterward.

D: Eu penso, prontos, eu, eu introduziria a partir de, do quadro numérico, não é. Mas à medida que vou fazendo o estudo dos limites, vou tendo situações no gráfico! E aí já posso aproveitar a relação que eu tive com o quadro numérico para o quadro gráfico (S12/1924-26).

D: I think that, ok, I, I would introduce through the, the numerical setting, you know. But while I am studying the limits, I am looking at the graphs! And then I can use the link that I had with the numerical setting to the graphical setting.

Frederico refuted David's argument asserting that the teacher must teach students to use graphs.



F: Não podemos, euh, euh, euh, recorrer a esse ... a esse refúgio de que os alunos não entendem a interpretação do gráfico. O problema é nosso se não, o aluno não entende, é porque nós não estamos a fazer o aluno entender a interpretação dos gráficos. (...) É esse problema que devemos ultrapassar. O aluno **deve** saber, de facto, interpretar o gráfico (S12/1929-36).

F: We cannot, er, er, er, seek ... refuge in the fact that students are not able to interpret graphs. It is our fault if the students do not understand them, it is because we do not help the student to understand interpretation of graphs. [...] This is the problem that we must solve. The students **must** be able, actually, to interpret graphs.

In this quote, Fredrico is going deeper into the discussion, positioning himself as a teacher who challenges the usual mathematics teaching in schools: if students are not able to work with a graph, it is because we, teachers, are not teaching graphs.

Mateus entered the discussion supporting Frederico's position.

M: Se estivesse a dar limites, eu optava de facto pelo método gráfico, porque... o, o método numérico, como o próprio David disse, disse que o aluno vê a aproximação dos números, só que de novo ficamos por aí, limites como simplesmente números. Então o método gráfico também permitirá entender, por exemplo, o conceito de limite por meio de recta tangente, porque ele deu o tal método, por exemplo se escolhermos este exemplo A., só o movimento no ponto Q, aqui na recta tangente. Então se eu, só falo de números, como é que ele [o aluno] vai interpretar o ponto Q como um movimento? E também fala aqui de velocidade instantânea. Como é que o aluno há-de ver que o tal número também está relacionado com o movimento? (S12/1944-52)

M: If I was teaching limits, I would choose the graphical method, because ... the, the numerical method, as David himself said, he said that the student can see the numbers approaching, but again we will stay there, limits only as numbers. Then the graphical method will also allow understanding, for example, the limit concept through the tangent line, because he gave that method, for example if we choose this example, the point Q movement, here in the tangent line. Then if I only speak about numbers, how could he [the student] interpret point Q as a movement? And also speaking about instant velocity. How will the student see that this number is also related to a movement?

Mateus had already spoken about using graphs during the first interview. He is now able to elaborate this idea, referring to two alternatives presented in Abel's dissertation: introduction through the tangent line, and introduction through instantaneous velocity. He then explained some teaching problems related to graphs, not only with limits but also with inequalities.

M: De facto tem havido esses problemas, eu assisto os meus colegas a, que estão a dar a 9ª classe no... inequações, não é, lineares. Alguns acabam não dando a resolução gráfica de inequações porque alegam que os alunos não entendem, mas então quem vai fazer entender? (S12/1956-)

M: In fact these problems exist. I see my colleagues who are teaching Grade 9 ... inequalities, you know, linear inequalities. Some of them do not teach the graphical resolution alleging that students do not understand, but then who will help them understand?

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In this quote Mateus shows that he deeply reflected on the use of graphs in secondary schools, not only to teach limits but also other mathematical topics. For him it is a more general issue.

As Abel was not participating in the discussion, I asked for his opinion. He stated:

A: Bom, o método gráfico é, acho que é o melhor... tanto mais que nós estamos sempre a pensar, euh, o conceito de limite em termos estáticos, números, cálculos. Bom, vamos tentar uma outra variante em termos de movimento (S12/1976-78).

A: Well, the graphical method is, I think it is the best ...all the more since we always think about, er, the limit concept in static terms, numbers, calculations. Well, we have to try another variant in terms of movement.

In this quote, Abel links the introduction of the concept with the different features (static or dynamic) of the limit: the limit concept is usually taught as a static number; graphs could show it in a more dynamic way. This is what he tried to do with his experiment, but he was disappointed with the results, as becomes illuminative during the 3<sup>rd</sup> interview. It may be why he did not engage much in the discussion.

Mateus then argued that with the numerical method, students become confused when they get to one-sided limits. Then David said that he was giving up the discussion, but that his opinions should be respected.

During this discussion, we can see that the teachers hold different positions.

Frederico and Mateus seem to have strong ideas about the way limits should be introduced in schools. They would start with graphical representations, because graphs are visual and using graphs would help students acquire a better concept image of limits. They refuted David's argument about the difficulties that students face with graphs: teachers do not teach graphs; it is why students face these difficulties. During this discussion, Frederico and Mateus showed that they have reflected deeply on the teaching of limits. They position themselves as teachers who can challenge the secondary school institutional relation to graphs. I classified their mathematical knowledge about the first encounter at the time of the 12<sup>th</sup> seminar as FE-MK2 and his ideas about teaching as FE-T5 (see pages 189-190).

David's position did not seem so strong. At the beginning of the discussion he stood up for a numerical introduction, because of students' difficulties with graphs. Then he said that after the numerical study, he would also introduce graphs, as suggested by his colleagues. At the end of the seminar, he gave up the discussion and seemed to get a little bored because Mateus and Frederico did not respect his point of view. As I already said, David was the youngest of the group, and had limited teaching experience. It was difficult for him to discuss the teaching of limits with older and more experienced colleagues. I classified his knowledge of the first encounter as FE-MK2 and FE-T4.

Abel did not enter the discussion until I asked for his opinion. He then agreed with Frederico's and Mateus's point of view, arguing that the graphical method gives an idea of movement. The limit concept is always taught from a static point of view. This assertion shows that Abel understood the different features of the limit concept, and the limitations of the usual way of teaching this concept in schools. However, as it was the topic of his dissertation, one would have expected him to engage more actively in the discussion. It seems that the results of his experiment drove him to lose his conviction that a graphical introduction using computers would help students understand the limit concept, as we will see in the next section. I also classified his knowledge as FE-MK2 and FE-T4.

The evolution of teachers' knowledge through my categories, from the first interview up to the 12<sup>th</sup> seminar, is visualised in Table 7. 4.

**Table 7-4 Evolution of teachers' personal relation to the first encounter – 12<sup>th</sup> seminar**

	<b>1<sup>st</sup> interview</b>	<b>Abel's 2<sup>nd</sup> interview and dissertation</b>	<b>12<sup>th</sup> seminar</b>
<b>Abel</b>	FE-MK1 FE-T2	FE-MK2 FE-T5	FE-MK2 FE-T4
<b>Mateus</b>	FE-MK1 FE-T3	N/A	FE-MK2 FE-T5
<b>David</b>	FE-MK1 FE-T1	N/A	FE-MK2 FE-T4
<b>Frederico</b>	FE-MK1 FE-T3	N/A	FE-MK2 FE-T4

## 7.5 Third Interview

### 7.5.1 Abel

At the beginning of the third interview, Abel spoke about the way he taught limits in schools. He was worried because he realized that he taught L'Hôpital's Rule before teaching derivatives.

Recordo-me que, bom dava exercícios sobre limites, euh ... principalmente na, eram funções polinomiais se não estou em erro, bom, para mi, o processo prático era, não é, utilizar aquilo que muitas das vezes chamamos de regra de L'Hôpital, porque era prático e [*suspira*] mas ... afinal de contas, agora vim a saber que, bom, como utilizar essa regra de L'Hôpital se o aluno ainda não deu derivada? E limites dá-se antes de derivada ... Mas ... eu vi que afinal de contas é um erro que eu estava a cometer na altura ... mas ... (I/A/81-87)

I remember that, well I gave tasks about limits, er ... mainly, they were polynomial functions I think, well, for me, the practical way was, you know, use what we usually call L'Hôpital's Rule, because it was practical and [*sighing*] but ... after all, now I got to know that, well, how could I use that L'Hôpital's Rule if students had not yet learn derivatives? And limits come before derivatives ... But ... I saw that after all I was doing a mistake at that time ... but ...

This quote clearly shows that Abel reflected on his teaching during the research process. He discovered that he was teaching in a way that students could not understand. In fact, L'Hôpital's Rule allows us to calculate the limit of the quotient of two continuous and derivable functions when it leads to an

indeterminate form such as  $\left[\frac{0}{0}\right]$  or  $\left[\frac{\infty}{\infty}\right]$ . For example:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{3x - 6} = \left[\frac{0}{0}\right] = \lim_{x \rightarrow 2} \frac{2x}{3} = \frac{4}{3}. \text{ In that way, the same rule can be used to}$$

calculate many indeterminate forms without using all the different techniques usually introduced in schools (see Chapter 2). It is not part of the secondary school syllabus because it involves derivatives, which are taught after the limit concept and at the end of Grade 12. I thus asked Abel how students got to use this rule before learning derivatives. He explained using the following example:

$$\lim_{x \rightarrow 2} \frac{x^3 - 4x^2 + x - 1}{x^2 - 3}. \text{ In fact he should not use this example because it is not an}$$

indeterminate form. Substituting  $x$  by 2 we just have  $\lim_{x \rightarrow 2} \frac{x^3 - 4x^2 + x - 1}{x^2 - 3} = -7$  but

Abel did not remember to check that. He calculated this limit on a sheet of paper using L'Hôpital's Rule and explained:

Então eu pegava esse 3, dizia, bom, os meus alunos, vamos subtrair, não é, euh, portanto 3,  $x$  ao quadrado menos, depois, euh, 4 vezes 2,  $8x$ , e aqui, onde não há, o expoente que é um nesse caso, não é, fica mais ... mais, e aqui, prontos, continuamos, não é,  $2x$  e acabou! Mas depois ... o aluno podia reter isso mas agora! ... se não tem o conceito de derivada! ... Quer dizer, aquilo era só encaixar e, prontos, o aluno lá resolvia assim porque tem que subtrair (I3/A/121-132).

Then I would take that 3 and say, well, my students, we are going to subtract, you know, er, I mean 3,  $x$  square minus, then, er, 4 times 2,  $8x$ , and here, where we don't have, the exponent is one in that case, you know, it remains plus ... plus, and here, ok, we go on, you know,  $2x$  and that's it! ... The student could keep that in mind but now! ... if he doesn't know derivatives! ... It means, he could only memorise and, ok, the student did it because he had to subtract.

On the sheet it reads:  $\lim_{x \rightarrow 2} \frac{x^3 - 4x^2 + x - 1}{x^2 - 3} = \frac{3x^2 - 8x + 1}{2x}$ , without the indication of

the limit in the second expression.

Abel taught this technique to calculate indeterminate forms for rational functions in a very algorithmic way. This technique could not be justified when he taught it because students did not have the relevant previous knowledge. It seems that he only became aware of this fact during the research process. His main concern as a teacher was that his students were able to calculate indeterminate forms very quickly, as he stated himself.

O problema é que, para o meu caso ... eu via aquilo tão prático, rápido, hum, processo rápido, então ... mas afinal estava a cometer um erro! (I3/A/151-56)

The issue is that, for me ... I saw that it was so practical, quick, hum, a quick process, then ... but after all I was doing a mistake!

Abel now realized that the way he introduced limits at school was inappropriate and that he did not know of other ways of introducing the limit concept.

Dava a definição! ... E acabou! Posto isto, dava os exercícios. Muito longe de pensar que, bom, essa coisa da introdução do conceito de limite, euh ... é um, um processo, é uma coisa, não só dar aquela definição porque o aluno, de facto, nem entendia, só escrevia lá no caderno, mas que afinal de contas para introduzir o conceito de limite, não é bem assim como eu fazia, não é, de que existem aquelas formas, os quadros que eu consegui ver no meu trabalho (I3/A/178-83).

I used to give the definition! ... And no more! Given that, I gave the tasks. I didn't think that, well, this issue of introducing the limit concept, er ... it's, its' a process, it's something, not only the definition because the student, in fact, did not understand, he wrote in his exercise book, but after all to introduce the limit concept, it is not what I did, you know, all these forms, these settings that I could see in my work.

In this quote it is very clear that Abel taught limits through the  $\varepsilon$ - $\delta$  definition and algebraic tasks, as a “good subject” of the institution. He now became aware of the consequences of this way of introducing limits for students’ understanding. However, prior to his research, he did not know of any other way of organizing students’ first encounter with limits. As already said, Abel is the only teacher in the group who taught limits in Grade 12. While the other teachers were reflecting on general practices in Mozambican institutions, his involvement was more personal because he also had to reflect on his own practice, which is much more challenging.

Later on during the interview, I asked him whether he could imagine at the beginning of the research that he would find all these alternatives to introduce the limit concept. He answered that he did not, and explained.

Não porque, é como eu disse, que eu dei aula ... claro que foi pouco tempo [em que ensinou limites] mas, é aquilo que eu disse ... dei a definição, prontos, dava as regras, vamos aos exercícios (I3/A/421-23).

No because, it’s as I told you, I taught ... of course it was a short time [of teaching limits] but, it’s what I said ... I gave the definition, ok, I gave the rules, we move on to the exercises.

From this quote it becomes clearer how Abel taught limits of functions at school. He probably began with limits of sequences, according to the syllabus, as he said during the first interview. He then introduced limits of a function of a real variable through the  $\varepsilon$ - $\delta$  definition (“I gave the definition”). Then the students were required to solve some algebraic tasks. It was how he learnt and he was reproducing this method of teaching limits, as he stated himself.

Bom, para mi também eu estava resumido a ... àquele conhecimento, assim, é aquilo que eu aprendi e também o manual é aquilo que dá, e eu vou transmitir [aos estudantes] (I3/A/428-29).

Well, I was myself reduced to ... to that knowledge, thus, it’s how I learnt and it’s also what the textbook shows, and I’m going to pass it on [to students].

From that quote it is clear that, despite students’ difficulties, and his own difficulties as a student and as a teacher, Abel did not challenge the institutional relation to limits prior to the research process.

As an experiment, Abel introduced the limit concept in a classroom, using the graphical register in a computer environment. He justified this choice as follows.

Na minha óptica, achei que seria uma forma ... fácil de os alunos puderem

In my opinion, I thought it would be a way ... an easy way for students to understand

entender... tanto mais que ... euh, representações gráficas de funções já vêm dando desde ... portanto, o capítulo de funções já vêm dando desde, digamos 8<sup>a</sup>, 9<sup>a</sup>, 10<sup>a</sup>, então eu achei bom! ... Porquê não? Mais um reforço! [...] talvez visualizando os alunos possam entender este conceito de limite, não é, através do... do quadro gráfico, utilizando um computador para eles puderem ver a imagem (I3/A/315-20).

... all the more since ... er, they studied graphical representations since ... I mean, the chapter on functions comes from, let's say grades 8, 9, 10, so I thought ok! ... Why not? Another follow-up! [...] maybe visualizing the students could understand this limit concept, you know, through the ... the graphical setting, using a computer for them to see the image. [...]

He had the idea that visualizing limits on graphs, using functions which were already familiar to students, could help them understand the concept. Using a computer would give them an opportunity to analyse many graphs in a short time and to manipulate these graphs using the computer's commands (I3/A/321-23). However, the results were not those that he was expecting.

A: É o que estava a imaginar que, bom, provavelmente assim os miúdos, ah! os alunos consigam entender a coisa. Mas, bom, tive os resultados que tive ... euh, não eram aqueles que eu esperava ... Bom! Mas eu estava, estava feliz a pensar que ia dar conforme eu tinha programado!

A: It's what I thought that, well, the kids would probably, I mean the students would understand the issue. But, well, I got the results that I got ... er, they are not what I was expecting ... Ok! But I was, I was happy thinking that it was running as I had planned!

I: Ficou decepcionado!

I: You felt disappointed!

A: Eh! Sim! (I3/A/323-30).

A: Eh! Yes!

Despite this disappointment, as his supervisors, my colleague and I tried to show him that this was an interesting result of his research: using graphs produced by a computer utility does not solve all difficulties that students face in understanding the limit concept. Even so he seemed to consider the results of his experiment more like a failure as a teacher than as a result to be analysed as a researcher. As I already mentioned, Abel seemed to feel rather insecure, and had difficulty taking a “step aside” to look at a teaching situation as a researcher rather than as a teacher. This may be because he taught himself during the experiment, playing the double role of teacher and researcher. In fact he was doing a kind of “action research”, reflecting on his own practice as a teacher while the other teachers were reflecting on institutional practices of teaching limits of functions. Moreover, he was reflecting not only on his early practice as a Grade 12 teacher, but also on his practice during the experiment, which was supposed to improve his former practice. It was a rather uncomfortable situation. This supports the conclusion

drawn in Chapter 5: teachers' research on institutional practices, instead of personal practices as in action research, would allow them challenge the institutional relation.

When I asked him how he would introduce the limit concept in schools, Abel stated that he would "try to use all the possible alternatives" (I3/A/839-40). He then worried about institutional constraints: the syllabus and the time allocated for this topic.

Agora precisaria talvez, atendendo, euh, o programa, eu não sei, agora quantas ... talvez continua o número 21, 21 horas ... Se seria possível com essas turmas enormes! (I3/A/846-48)

Now I would need maybe, er, considering, er, the syllabus, I don't know, how many now ... maybe it's still 21, 21 hours ... If it would be possible with these huge classes!

From this quote it is quite clear that Abel knows the syllabus, even how many hours are dedicated to the study of this topic, and the secondary school context (huge classes). He then concluded that, considering these institutional constraints, it would be difficult to try to use all alternatives.

Eu havia de utilizar, vamos supor primeiro dia de introdução, vamos ao quadro gráfico, euh, depois amanhã vamos tentar o numérico, com gráfico, vamos tentar, portanto ... euh ... geométrico, mas depois? (...) Mas então, isso para dizer que a pessoa pode ter aquela vontade sim, vamos testar todos [as alternativas], mas ... o contexto em si, vamos, euh, euh, vou conseguir, com turmas enormes? A não ser que vá roubar uns Sábados, não sei! É isso que estou a, sozinho a tentar ver (I3/A/852-65),

I would use, let's say on first day of introduction, we go to the graphical setting, er, then tomorrow try the numerical, with graphs, try, I mean ... er ... geometrical, but after that? (...) But now, it's to say that you can have that intention yes, I will test all of them [the alternatives], but ... the context itself, will we, er, er, will I be able, with huge classes? Unless I take some Saturdays, I don't know! That's what I've, alone been thinking about,

In this quote, Abel was speaking about using all alternatives for introducing the limit concept, one after the other, with the same students, as did Mateus during the 12<sup>th</sup> seminar (see page 213). This does not make sense as they are "alternatives". He could think about using different alternatives with different students and then compare the results, but not all of them with the same students. As observed before, doing that would be time consuming and probably confusing for students. Does Abel understand the alternatives for introducing the limit concept that he presented in his dissertation?



To sum up, during the third interview it became clear that prior to research Abel taught limits in a very algorithmic way, through the  $\varepsilon$ - $\delta$  definition and a rule (L'Hôpital's Rule) that students could not understand because it involves derivatives. This confirms the classification made of his prior knowledge (FE-MK1 and FE-T1). He is now aware of this fact and worried about it. He collected several ways of organising students' first encounter with the limit concept from a few textbooks, but had difficulties in analysing them in terms of students' learning. He was then strongly convinced that introducing limits through graphs and using a computer environment would help students understand the concept (FE-T5). Nevertheless, his experiment did not meet his expectations and he felt disappointed. He was now willing to try other ways of introducing limits but does not seem to have a clear idea of how to do it. He does not seem to be able to analyse the results of his experiment in order to suggest another way of organising the first encounter. Therefore I classified his knowledge at the end of the research as FE-MK2 and FE-T4 (see pages 189-190).

### 7.5.2 Mateus

During Mateus' third interview, we did not specially focus on the different ways of introducing limits of functions, but Mateus explained how he would teach this topic at secondary school level. He said that he would start by giving everyday examples.

Euh, a primeira coisa que vou é... procurar diferentes maneiras de introduzir o, o conceito. Portanto a partir por exemplo da própria vida prática, não é! (I1/M/945-47)

Er, the first thing that I will ... I'll look for several ways of introducing the, the concept. So starting from for example from the real life, you know!

Then, to introduce limits of functions, he would use several methods, starting from the numerical register.

Havia de escolher um, um método, e eu acho que não havia só de me cingir numa coisa Então, para caso de limites de funções, eu havia de, de começar pela definição intuitiva. Sim, a partir dum quadro numérico (I1/M/989-91).

I would chose a, a method, and I think that I would not use only one thing. So, for limits of functions, I would start with the intuitive definition. Yes, starting from the numerical setting.

It seems that Mateus changed his mind since the 12<sup>th</sup> seminar. During that seminar, he refuted David's idea of starting with the numerical register, arguing

that “we will stay here, limits only as numbers” (S12/1946). He now states that he would start using numerical examples, and specifies that he would do that for several kinds of functions.

... procurar substituir valores que não só a função tenha limite, mas também o caso em que a função não tem limite ... Porque ... alguns manuais, euh, os exemplos que se dão para a introdução do, de limite, é só as funções que têm limite ... Então fica difícil para as funções que não têm limite (...) como professor, começo por uma função que tem limite e uma outra função que tende, que não tem limite mas a partir dum quadro numérico (I3/M/996-1011).

... trying to substitute values not only when the function has a limit, but also when the function doesn't have a limit ... Because ... in ... some textbooks, er, the examples used to introduce the, the limit, it's only functions which have a limit (...) as a teacher, I begin with a function which has a limit and another function which tends, which doesn't have a limit but starting with the numerical setting.

Here, Mateus made an interesting remark about the kind of functions that he would use to introduce the limit concept. Most books only use numerical examples of functions when the limit exists. He argued that it is also necessary to present other cases. In fact, if students are only faced with one kind of situation (the limit exists), they could assume that it is always the case. Mateus is aware of this possible deviation.

He said that he would then introduce a graphical interpretation.

E depois, já para interpretar, então posso dar o... o método... gráfico (...) porque eu acho que o gráfico é mais marcante (I3/M/1015-20).

And then, to interpret, then I can use the ... the graphical method (...) because I think that the graph has more impact.

He would lead students to calculate limits and then use the results to sketch the graph of the function.

No método algébrico, eu havia de, o que vou fazer, não digo havia, o que vou fazer é procurar... que, a partir, procurar que os alunos constroem gráficos a partir do limite. E não a partir dos gráficos procurar os limites. A partir dos gráficos procurar limites é o que vou dando lá na, na interpretação, no ... nos primeiros, nas primeiras aulas (I3/M/1027-1035).

With the algebraic method, I would, what I'm going to do, I don't say I would, what I am going to do is to try ... that, from, lead students to sketch graphs from the limits. And not starting from the graph read the limits. Reading the limits from the graphs it's what I would give during the, the interpretation, during ... during the first, the first classes.

Mateus would ask students to work with graphs in two ways. The first one is reading limits from graphs, at the very beginning of the work with limits. Later he would lead them to sketch the graph of a function using its limits, which is a much more difficult task.

He also said that he would try to show the importance of limits to his students (I3/M/1035-1047), as well as speak about the history of this concept (I3/M/1051-1054), showing that he was aware of the necessity of justifying the teaching of this concept in secondary schools, as we will see in Chapter 8.

Then Mateus said that he would give the students three definitions. The first one would be the more intuitive one<sup>17</sup>, “for the student to have an idea of what a limit is” [para o aluno ter uma certa ideia do que é limite] (I3/M/1091-2). Then he would give the definition by Heine (see footnote page 192) (I3/M/1118). Finally the third one would be the  $\varepsilon$ - $\delta$  definition.

Agora essa definição, a definição 2, 3, que é a definição formal... posso dar... mas depois dos alunos, tanto interpretado o limite, e tanto se exercitarem de limites, então aqui muito mais para, euh, uma questão, vou dizer que uma questão formal. [...] Então a terceira definição para mim seria uma definição já final, depois do próprio aluno entender o significado do, de limite e o próprio aluno pelo menos dando-me por suas próprias palavras do que entende por limite (I3/M/1136-54).

Now this definition 2, 3, which is the formal definition ... I can give it ... but after students having interpreted a lot of limits, after training limits a lot, then much more for, er, a question of, I would say something formal. [...] So for me the third definition would be the final one, when the student understands the meaning of, of limit and the student himself can explain with his own words what limits mean for him.

Mateus argues that the  $\varepsilon$ - $\delta$  definition of limits only can be understood by students having a good concept image of limits, after having worked with limits in an intuitive way.

To sum up, it seems that at the end of the research process Mateus has strong ideas on how to organise students’ first encounter with the limit concept, linking it with the everyday concept of limits, using different representations, applying limits to sketch graphs, and giving several definitions. His explanations show that he is aware of students’ difficulties and of possible teaching deviations (using the same kind of functions). Mateus’ personal relation to the first encounter with limits seems to have evolved since the 12<sup>th</sup> seminar. He is now able to explain how he would like to introduce limits at secondary school level in a quite elaborated way, using different representations (his own topic) and taking into

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<sup>17</sup>  $\lim_{x \rightarrow a} f(x) = L$  means that  $f(x)$  approaches  $L$  as  $x$  approaches  $a$

account the history of the concept (Frederico's topic), and its importance (David's topic). Furthermore he seems to be rather confident in his opinions. For these reasons I classified his knowledge as FE-MK2 and FE-T6 (see pages 189-190).

### 7.5.3 David

During the third interview, David said that the alternatives to introduce limits were a “disturbing topic” (“tema conturbado”). He explained.

É muito conturbado porque (...) quando fala muito de ... diferentes alternativas de introdução ... não fugimos muito daquilo que é a definição formal do próprio limite (I3/D/1999-2008).

It's very disturbing because (...) when speaking a lot about ... different alternatives of introduction ... we don't escape much from the formal definition of limits.

Despite his difficulties in understanding the  $\varepsilon$ - $\delta$  definition of limits, David seems to consider its teaching as essential, even at secondary school level. He does not seem to have a clear memory of the several ways of introducing the limit concept that Abel presented during the seminars. He vaguely remembered hearing about some of these alternatives but did not have a clear idea about them.

Recordo-me que num, num dos seminários ele [Abel]... falou de que, através da, da, da própria Física, não é, podíamos introduzir pela velocidade instantânea... Eu recordo que ele falou... prontos essa é, é uma das alternativas que sempre pode utilizar, pode-se utilizar também... utilizando um quadro, não é, ele também chegou a expor... Não estou, estou um bocado vago mas... (I3/D/2012-20)

I remember that, during one of, of the seminars ... he [Abel] said that, through the, the, the physics itself, you know, we could introduce through the instantaneous velocity ... I mean this is one, one of the alternatives that can be used, we could also use ... using a table, you know, he also explained ... I don't, I am rather vague about it but ...

Nevertheless he got new ideas on the way the limit concept should be introduced and tried to explain them: he would introduce the limit concept using, in the first place, the numerical register.

Como é que faria? ... Com a função, eu iria pelo quadro numérico! ... Eu usaria o quadro numérico! ... Porque eu acredito que com o quadro numérico, facilmente eu posso explicar o que é limite, qual é a ligação disso com o termo “aproximação”... Agora mesmo com a parte da, da, da Física, não é, eu acredito que também é possível introduzir-se com base nisso (...) Mas eu acho que talvez o quadro numérico seria o mais abrangente (I3/D/2033-61).

How would I do? ... With the function, I would go to the numerical setting! ... I would use the numerical setting! Because I believe that with the numerical setting I could easily explain what is the limit, what is its link with the term “approximation” ... Now even with the, the, the physics part, you know, I believe that it is possible to introduce limits in that way (...) But I think that the numerical setting would be broader.

In this quotation, David reiterates the idea expressed during the 12<sup>th</sup> seminar, where he stood up for the numerical representation, against Frederico and Mateus' points of view. He is now able to elaborate a little more this idea: the numerical register allows us to evidence how the function approaches the limit.

I then asked him what he would do after using the numerical register. He said that he would teach the  $\varepsilon$ - $\delta$  definition, because any concept must have a definition, even if the students do not understand it.

Não havia de fugir à regra (...) Porque é um conceito, e todo o conceito tem que ser definido ... mas, eu particularmente acho que haveria de usar muito pouco (...) Eu havia de dar porque, prontos, é aquele questão de situação formal... Todos devem seguir o, pelo menos o conceito em si, não é... a definição e ... e para mi pessoalmente eu acredito também que não vejo em quê é que essa definição pode ajudar (I3/D/2069-2108).

I would not escape from the rule (...) Because it's a concept, and any concept must be defined ... but, personally I think that I would use it very little (...) I would give it because, you know, it's a question of formal situation ... Everyone must follow the, at least the concept itself, you know ... the definition and ... and for me personally, I also believe that I don't see how this definition can help [students].

This statement shows the weight of the institutional relation to a concept. Despite his own experience of learning a definition that he did not understand, even during his university studies, David would teach it at secondary school level, he “would not escape from the rule” or, in other word, he would be a “good subject” of the institution.

He also stated that the graphical register should be more explored in secondary schools and at university.

[o uso de gráficos] é um assunto que, no meu ponto de vista, deve ser explorado (...) são poucos os professores que gostam de ir por um, por um método gráfico, porque prontos, ai, há mais facilidade do próprio professor engasgar-se... e não ter muita saída de que o próprio aluno (...) por isso que muitos professores não optem por esse lado, optam sempre por um quadro (...) analítico que é o mais simples em resolver, o aluno fica simplesmente com aquela ideia de que limite é simples! (I3/D/270-301)

[using graphs] is an issue that, from my point of view, needs to be explored (...) few teachers like to use a, a graphical method, because, you know, then, it's easier for the teacher to choke ... and become as stuck as the student himself (...) that's why many teachers do not choose this side, they always choose an analytical setting (...) where it's easier to solve, then the student gets that idea that limits are simple!

In this quote, David analysed quite clearly what usually happens in secondary schools. Teachers do not feel comfortable with graphs. They are scared of not being able to explain properly to students the link between limits and graphs. This

is why they avoid working with graphs and only give students algebraic tasks and specific algorithms to solve them. In that way limits seem simple, even if students do not grasp the meaning of the calculations. It seems that David reflected about this issue since the 12<sup>th</sup> seminar, and he now uses some of his colleagues' arguments about teaching graphs in schools.

To conclude, I would say that David's knowledge on the different ways of organising students' first encounter with limits has evolved considerably since the first interview. At the beginning of the research, his knowledge was limited to algebraic procedures and the  $\varepsilon$ - $\delta$  definition that he did not understand (FE-MK1). Even if he does not have a precise knowledge of the alternatives presented in Abel's dissertation, he is now able to suggest a more elaborated way of introducing limits, and to challenge some aspects of the institutional relation to mathematics, especially the teaching (or lack of teaching) of graphs. Nevertheless, as a "good subject" of the institution, he would teach the  $\varepsilon$ - $\delta$  definition. I classified his knowledge as FE-MK2 and FE-T5 (see pages 189-190).

#### **7.5.4 Frederico**

During the third interview, we did not specifically address the issue of the first encounter with the limit concept, but Frederico said that, through the history of this concept, which was his own research theme, he was able to see that this topic must be handled carefully in order for students to get an idea about the concept itself (I3/F/36-39).

He stated that this concept was very abstract and therefore it was necessary to use several alternatives and methods for students to get an idea of limit (I3/F/40-42).

He then suggested that the study of limits should begin earlier than Grade 12.

Até talvez, na minha ideia até, euh, mesmo que fosse uma, uma questão de dar um, uns pequenos conceitos sobre, euh, limite a partir da 10<sup>a</sup> classe, em vez de começarmos já na, na 12<sup>a</sup>, ou mesmo na 11<sup>a</sup>. Mesmo que seja, euh, aquele, o uso de umas funções simples, mas para poder preparar portanto o, o estudantes ter uma ideia (I3/F/49-53).

Even maybe, it's my idea, er, even if it was a, a question of giving a, small ideas about, er, limits since Grade 10, instead of beginning in, in Grade 12, or even Grade 11. Even if it was, er, using simple functions, but to prepare, I mean the, the students to get some idea.

Even if he did not elaborate this idea, Frederico seems to be aware that the introduction of limits in schools should be progressive, starting with more intuitive work (“giving small ideas” and “using simple functions”). It is what he tried to do when he taught limits at the Agriculture School. We already saw that this was a good experience for him (see page 203). He then said that teachers only use the algebraic setting because they do not have much time, showing that he was aware of the institutional constraints (I3/F/82-88). He also stated that he would use a more graphical method to introduce limits (I3/F/336-37).

Speaking about Abel’s work, he added.

Aí eu gostei de que, o, porque aí de facto está envolvido, euh, duas coisas. Por outro lado estamos a falar de o próprio, de, de parte numérica, e, do outro lado estamos a observar directamente o, o gráfico. Aí acho que o, o aluno facilmente ele pode entender, ter uma ideia o que é isso de “tende para” ... (I3/F/341--48)

I liked that, because in fact it involves the, the numerical part, and, on the other hand, we are directly observing the, the graph. Then I think that the, the student can easily understand, can have an idea of what “tends to” means ...

From that quote, Frederico seems to be willing to use both graphical and numerical representations to give meaning to the limit concept. As in Mateus’ case, it seems that he reflected on this issue since the 12<sup>th</sup> seminar and now agrees in using both numerical and graphical representations. When I asked him how he would introduce limits in schools, he said that he would begin with figures, intuitive ideas coming from the history of the concept (I3/F/793-96). This choice is clearly influenced by his research topic. He then explained how he would introduce the limit concept in an intuitive way through a task.

Faria portanto um trabalho, assim um trabalho como um, euh, como é que poderia chamar isso, como um pré-requisito... Sim. A, usar aquelas ideias, euh, intuitivas que aparece sobre, euh, sobretudo aqueles de Arquímedes... e como, de, da, das figuras, euh, dos polígonos inscritos numa circunferência, euh, aquela, aquele também de reduzir o, o, o, o quadrado em quadradinhos cada vez mais pequenos (I3/F/801-13).

I would thus prepare a task, a task as a, er, I could call it, as a prerequisite ... Yes. Using those ideas, er, intuitive ideas that appear, er, especially those ideas of Archimedes ... and like, of, of, of figures, er of polygons inscribed in a circle, er, this, also this one to reduce a, a, a, a square in smaller and smaller squares.

These examples come from his dissertation. He said that he would give this task for students to solve in small groups. Then he would use other registers.

E depois é que já entrava na, no, no, nas  
próprias regras que são o uso de, de  
gráficos ... assim como... euh, o uso  
numérico e, e algébrico (I3/F/848-52).

And then I would enter the, the, the rules  
themselves which are the use of, of graphs  
... as well as ... er, the numerical and, and  
the algebraic [registers].

All these quotes are indicators of the evolution of Frederico's personal relation to the first encounter with limits at secondary school level. At the beginning of the research, he had an experience of teaching limits out of the syllabus in an intuitive way. He already had the idea that visualizing limits through graphs could help students understand this concept (FE-T3). He now is able to elaborate more how to organise students' first encounter with limits, using the results of his own research about the history of the limit concept, some results of Abel's work, as well as the discussion with his colleagues during the 12<sup>th</sup> seminar. I classified his knowledge as FE-MK2 and FE-T5 (see pages 189-190).

## 7.6 Conclusions

Several conclusions can be drawn from the analysis presented in this chapter.

In the first place, the stories of each teacher's contact with the limit concept during the first interview, and their view on their own prior knowledge on limits during the 2<sup>nd</sup> (for Abel) and 3<sup>rd</sup> interview, can be compared with my previous analysis of Mozambican didactic institutions' relation to this concept through the analysis of several documents (see Chapter 2).

In the second place, teachers' comments on the different ways of introducing the limit concept during the third interview and the discussion during the 12<sup>th</sup> seminar, contrasted with their comments during the first interview and, classified according to my categories, show what they learnt about these different approaches and the new ideas they got on how they would like to organise their teaching in a secondary school.

Finally, in these comments we can track the influence of their own topic in this learning. These conclusions will be presented separately.

### 7.6.1 Mozambican didactic institutions' relation to the limit concept

The way all four teachers described their contact with the limit concept through Mozambican institutions (and even through a German institution in Abel's case)



gives an image of the institutional relation to limits consistent with my previous study (see Chapter 2).

In secondary schools, the first local mathematical organisation  $MO_1$ , the algebra of limits, is dominant. Students are required to calculate limits using algebraic techniques. From teachers' comments, we can surmise that few technological elements are present to justify these techniques. An extreme case is the teaching of L'Hôpital's Rule before teaching derivatives, which only can be done through an algorithm that students are not able to understand. The only trace of the second mathematical organisation  $MO_2$ , which relates to the existence of limits, sometimes introduced in secondary schools is the  $\varepsilon$ - $\delta$  definition, which none of the teachers involved in the group understood. It is probably the case of many Mozambican mathematics teachers. As a consequence of this algorithmic way of teaching limits, students held a poor image of limits. For them it is merely the application of algebraic rules.

At university (EMU former Faculty of Education, Pedagogical University, and even a German University) the knowledge block is a little more elaborated than in secondary schools: it involves not only the  $\varepsilon$ - $\delta$  definition but also proofs using this definition. However, even at university level, teachers remember learning the  $\varepsilon$ - $\delta$  definition without being able to understand it. They had to memorise this definition and were sometimes requested to apply it, which resulted in many difficulties. Frederico remembered the teacher giving proofs that the students did not understand. In fact they were only required to calculate limits. Three teachers confirm that they mainly used two textbooks, by Demidovitch and by Piscounov.

Two teachers related an attempt to give meaning to the limit concept in two different institutions. On the one hand, Abel remembered that, in his teacher training course at EMU, the lecturer chose numerical values to approach the  $x$ -limit and asked the students to use them to determine the function's limit (see section 7.2). However he did the required calculations without understanding their meaning. On the other hand, David said that in his secondary school they used some graphs at the beginning to illustrate limits (see page 200). However afterward they only did calculations.

These procedures are quite common in Mozambique. During the introduction of limits of sequences, teachers usually give some numerical examples and some graphical illustrations of limits. These examples are seldom followed by numerical or graphical tasks to be solved by students. Students' tasks are algebraic, and they usually concentrate on what they need to solve these tasks. For this reason they do not deeply reflect on the numerical or graphical examples given by the teacher and, consequently, are not able to use the numerical and graphical registers as a means to evaluating or interpreting a limit. These examples are insufficient to help students construct a strong concept image of limits. For this to happen, they should be asked to independently solve numerical tasks, and independently provide a graphical interpretation of limits, in several different cases. Furthermore they should be accustomed to using the numerical and graphical registers as resources to evaluate or interpret a limit as shown in Chapter 3 (pages 70-72). Otherwise the use of numerical or graphical register remains something that the teacher does but most students do not understand.

As a conclusion, we can see that in these didactic institutions the same dichotomy exists between the merely algebraic practical block belonging to  $MO_1$  and the knowledge block belonging to  $MO_2$ , never used in practice. As a consequence, these teachers did not grasp the meaning of the limit concept. This institutional relation was the starting point of all four teachers' personal relation to limits (FE-T1), although some of them were critical about it (FE-T2).

### **7.6.2 Evolution of teachers' knowledge on the first encounter**

There is evidence that all four teachers' personal relation to first encounter with the limit concept evolved through the research process. However this evolution was different from teacher to teacher, due to their previous contacts with this topic, their own research topic and other personal factors. I will analyse them separately.

#### ***Abel***

At the beginning of the research Abel only knew how to introduce limits in schools as stated in the Mozambican syllabus (FE-MK1). As a student, he experienced another introduction to this concept at the Faculty of Education, using

numerical values, but did not understand it. As a consequence, he turned to memorization. As a teacher, he taught the  $\varepsilon$ - $\delta$  definition and algebraic rules to calculate limits, including L'Hôpital's Rule before derivatives, and was aware that his students had difficulties with this concept (FE-T2). He now realises that this was not a good approach and feels uncomfortable about this.

Through his own research he was able to find out several other methods to introduce the limit concept at secondary school level, and he presented them in his dissertation (FE-MK2). However he was not able to analyse in greater depth the relations between these introductions of the limit concept and students' understanding and ability to give meaning to the concept. Before his experiment in a secondary school, he was convinced that introducing limits through graphs, using a computer utility, would help students understand the limit concept (FE-T5). However, he was disappointed with the results of this experiment. He is now worried about how to apply these methods, considering the institutional constraints of Mozambican secondary schools (FE-T4). It is probably for this reason that, during the 12<sup>th</sup> seminar, he did not engage in the discussion about the best method to be used in schools until I asked for his opinion. A summary of the evolution of Abel's personal relation to the first encounter with limits can be found in Table 7.5 (page 236).

Abel was in a very different position from his colleagues within this topic. In the first place, he had taught limits in Grade 12 and was now reflecting on this teaching. This could be an advantage, because of knowing the topic better, as well as the syllabus, and the way it is taught in schools. However challenging his own teaching is more awkward than challenging an institutional relation. On the other hand, "Alternative ways of introducing the limit concept" was his own topic and he obviously had to learn a lot about it for his dissertation and to present it to his colleagues at the seminars. Nevertheless, he did not seem to understand that the several alternatives that he presented were "alternatives" and seemed willing to use them all at the same time. It was also difficult for him to analyse the results of his experiment.

Furthermore, as suggested before, Abel appeared to be a rather unconfident person. He was the more experienced teacher within the group, and he had to question his teaching of limits as a secondary school teacher, as well as his experiment that he considered as a personal failure as a teacher. This was very challenging for him, and could explain a kind of retrogression of his ideas about how to organise the first encounter with the limit concept at the end of the research process.

### ***Mateus***

Mateus' first contact with limits of functions was mainly algebraic. At the end of the first interview, he was able to criticise this kind of approach, possibly influenced by the tasks presented to him during the interview, and suggest a more graphical approach (FE-T3). His research about "Different settings and models to study limits of function in schools" reinforced these ideas on how to introduce limits at school level. During the 12<sup>th</sup> seminar, Mateus showed that his knowledge and views on the introduction of limits in a secondary school evolved a lot during the research process (FE-T5). During the third interview he explained that he would like to begin with everyday examples, then use several different representations (numerical, graphical and algebraic) and give three definitions. It seems that he consulted several textbooks and compared how they introduce the limit concept. He was worried by the fact that they always use the same kind of tasks (when the limit exists).

At the end of the first interview Mateus stated that he would use the numerical register and graphs to introduce the concept. It is interesting to note that he expressed this in very general terms. During the third interview he was able to better articulate how he "will" (and not "would") use different representations (numerical, graphical, and algebraic) and to explain his choices, going beyond the institutional relation (FE-T6). A summary of the evolution of Mateus' personal relation to limits regarding the first encounter can then be found in Table 7.5. (see page 236).

**David**

David studied limits at school and at university in a very algorithmic way, according to the usual institutional relation. He is now aware of this fact and even of the reasons why teachers introduce limits in that algorithmic way: they themselves do not feel comfortable when working with graphs.

David does not remember all the alternatives to introduce limits that Abel presented during the seminars but, as Mateus, he acquired new ideas about how to introduce limits in a secondary school. During the 12<sup>th</sup> seminar, he suggested a numerical introduction, but faced difficulties in refuting Frederico and Mateus' arguments in support of a graphical introduction (FE-T4). During the 3<sup>rd</sup> interview, he presented a more elaborated idea about the first encounter, using both numerical and graphical registers. These new ideas probably developed through the discussions about Abel's work during the seminar. However he did not challenge the institutional relation to limits: he would teach the  $\varepsilon$ - $\delta$  definition, even knowing that students can not understand it (FE-T5). A summary of the evolution of David's personal relation to limits regarding the first encounter can also be found in Table 7.5 (next page).

**Frederico**

Within two Mozambican institutions where Frederico studied limits, this concept was introduced in a rather theoretical way that the students did not understand (FE-T2). As a teacher, he had a better but very short experience of teaching limits of sequences in a more intuitive way, using the numerical register. At the end of the 1<sup>st</sup> interview, he suggested that limits could be introduced through graphs (FE-T3). Through the research process he became aware that this topic should be introduced using different settings and registers, in particular the numerical and graphical registers. He is willing to use them at secondary school level, starting with some historical tasks coming from his own work (FE-T5). See Table 7.5 for a summary of the evolution of Frederico's personal relation to limits regarding the first encounter.

### Overview

The evolution of the four teachers' personal relation to the first encounter with limits is summarised in Table 7.5.

**Table 7-5 Evolution of teachers' personal relation to the first encounter**

	1 <sup>st</sup> interview	Abel's second interview and dissertation	12 <sup>th</sup> seminar	3 <sup>rd</sup> interview
<b>Abel</b>	FE-MK1 FE-T2	FE-MK2 FE-T5	FE-MK2 FE-T4	FE-MK2 FE-T4
<b>Mateus</b>	FE-MK1 FE-T3	N/A	FE-MK2 FE-T5	FE-MK2 FE-T6
<b>David</b>	FE-MK1 FE-T1	N/A	FE-MK2 FE-T4	FE-MK2 FE-T5
<b>Frederico</b>	FE-MK1 FE-T3	N/A	FE-MK2 FE-T5	FE-MK2 FE-T5

This table shows the following trends:

- At the beginning of the research, all teachers knew how the limit concept is usually introduced in Mozambican didactic institutions:  $\varepsilon$ - $\delta$  definition, rules and tasks to calculate limits (knowledge of the institutional relation – FE-MK1).
- Three of them (Abel, Mateus and Frederico) expressed their worry that this kind of first encounter does not help students understand the limit concept (FE-T2), Abel during the first part of the interview, Mateus and Frederico during the second part.
- At the end of the first interview, two of them (Mateus and Frederico) suggested that a more graphical approach could help students understand the concept (FE-T3). Their suggestions may have been influenced by the tasks shown during the interview.
- At a first stage of his research about “Alternatives ways of introducing the limit concept” Abel reviewed several ways of organising the first encounter (FE-MK2) and was convinced that he had found a good method

to introduce the concept (FE-T5). He felt disappointed with the results and is no longer sure about how to introduce limits (FE-T4).

- During the 12<sup>th</sup> seminar, Mateus and Frederico showed strong conviction that a graphical introduction would be better for students (FE-T5), while Abel and David defended their positions without much conviction (FE-T4).
- During the 3<sup>rd</sup> interview, David appeared to have stronger convictions about ways of introducing limits (FE-T5). Mateus also had stronger convictions and could anticipate some problems such as always using the same kind of function and how to avoid these problems (FE-T6). Abel and Frederic's position did not show much change (FE-T4 and FE-T5, respectively).

Even if they are different, the personal relation to limits of the four teachers was closer to the relation of the new institution at the end of the research project. The differences between teachers can be explained in several ways.

As explained before, Abel was in a different position from his colleagues, having to reflect and challenge his own teaching and research process, in particular his experiment. It was a more uncomfortable position, which could have hindered the evolution of his personal relation to limits regarding this issue.

Abel's difficulties in challenging the content of his teaching help us reflect on the teachers-as-researchers movement. As showed in Chapter 5, teachers usually choose to research their own practice or students' difficulties, taking for granted the institutional relation to mathematical knowledge and, therefore, their own personal relation to mathematics. In the new institution, Abel, who is an experienced teacher, had to reflect on the content of his own teaching, which was very hard for him. He taught limits for several years, and now became aware of the limitations of his own knowledge of this topic. This is much more difficult than challenging ways of teaching, because a teacher is supposed to master the knowledge to be taught. It is more acceptable that he finds difficulties in teaching that knowledge.

During the whole research process, Mateus showed stronger mathematical knowledge than his colleagues. He was also the most active in consulting books and giving references that could be interesting for his colleagues' work. This can explain why he was able to go beyond the institutional relation.

David was the youngest of the group and had little teaching experience. This can explain why, during the first interview, he only mentioned his own difficulties with the limit concept and did not mention students' difficulties in general. He was putting himself in the student's position and not in the teacher's one. It can also be the reason why he had difficulty in defending his own ideas about the first encounter during the 12<sup>th</sup> seminar.

Frederico's mathematical knowledge was not very strong, and he sometimes confused concepts, for example limit of functions and convergence of sequences (see first interview). He appeared to be a very calm and analytic person, and was always the first one to analyse his colleagues' work during the seminars. These qualities helped him overcome some of his mathematical difficulties.

These differences in the evolution of teachers' knowledge of limits of functions during the research process raise the following question: For whom might research on mathematical contents be more beneficial? For teachers in training? For teachers in their early years of teaching? For experienced teachers? Abel's difficulties in challenging the content of his teaching during the research process suggest that this kind of research can be awkward for experienced teachers. This point will be discussed further in the following chapters.

In addition to these personal differences between teachers, the topic of their individual research may also have influenced their learning about the first encounter. I analyse this aspect of the evolution of their knowledge in the next section.

### **7.6.3 Influence of teachers' personal topic on their learning about the first encounter**

Some of the teachers' utterances during the interviews and the 12<sup>th</sup> seminar reveal the influence of their own topic on their learning about the first encounter. I will



not focus on Abel's case, whose research was exactly on this topic and obviously influenced his personal relation.

Mateus' topic was "Different settings and models when teaching the limit concept". He referred to this topic already during the first interview, when speaking on "algebraic model". During the third interview, he also referred to the "numerical register", and also to his colleagues' topics: the importance of limits (David's topic) and the history of the concept (Frederico's topic). This shows that he was able to link all the topics in order to figure out a new way of teaching limits.

David's topic was "Applications of the limit concept in Mathematics and in other sciences". He did not refer to his own topic during the discussion about the first encounter. During the seminar and the third interview, he used the word "setting" several times, showing that this concept became familiar to him.

Frederico's topic was "The history of the limit concept". He referred to this topic during the third interview, saying that he would begin with a task coming from the history.

The influence of their own topics is apparent for Abel, Mateus and Frederico, but not for David.

In this chapter, I showed that the evolution of teachers' knowledge about the first encounter with the limit concept through the research process has been uneven, depending on their previous contacts with this concept and of their own research topic. These results are important for analysing the teachers-as-researchers movement, as well as the possible benefits of research on the institutional relation to mathematics, depending on the experience of the teachers involved.

The next chapter provides an analysis of the evolution of the teachers' knowledge about the social justification for teaching this topic at secondary school level.

# **CHAPTER 8**

## **THE SOCIAL JUSTIFICATION FOR TEACHING LIMITS IN SECONDARY SCHOOLS**

## **8 The Social Justification for Teaching Limits in Secondary Schools**

In the previous chapter, I presented a detailed analysis of the evolution of teachers' personal relation to one aspect of *mathematics for teaching* (MfT) limits: how to organise students' first encounter with the limit concept. I concluded that their personal relation to limits regarding the first encounter appeared to have noticeably evolved, albeit unevenly.

In Chapter 3, I argued that a teacher should be aware of the social justification for teaching specific content in a specific moment of the syllabus. I then argued in Chapter 4 that the justification for teaching limits of functions at secondary school level was mainly because it is a very strong concept: it is the basic concept for differential and integral calculus, and it has many applications in other sciences. Students should learn the limit concept in secondary schools because they will need it to understand the concepts of derivative and integral in their further mathematical studies, and they will need it in other subjects, such as physics and biology. The topic of David's dissertation, "Applications of the limit concept in mathematics and in other sciences", directly addressed some of the reasons that legitimate the teaching of this topic in secondary schools, and his findings were presented and discussed during the several seminars (see content of the seminars in Appendix 6.7).

In this chapter, I present the results of the evolution of teachers' knowledge about the social justification for teaching limits at secondary school level. As explained in the methodological chapter (see Chapter 6), while the analysis has been done using the same procedures, I will not provide as detailed an analysis in this chapter as in the previous chapter.

This chapter is structured as follows:

- 8.1. Data collection and analysis
- 8.2. David
- 8.3. Abel

8.4. Mateus

8.5. Frederico

8.6. Conclusion

## **8.1 Data collection and analysis**

Data used for analysing the evolution of teachers' knowledge of the social justification for teaching limits derive from the first interview (teachers' personal relation to limits prior to research) and the third interview (final personal relation).

During the first interview, the social justification for teaching the limit concept was addressed through questions such as:

- In secondary school, did you understand which applications used the limit concept?
- Did you understand its applications at university better? How? Why?
- In your opinion, what kinds of applications of the limit concept should be taught in schools?
- Do you think that it is useful to teach limits of functions in secondary schools? Why? How do you think that students will use this concept later, during their studies at university for example? In which subjects? In which areas?
- Do you think that “limits of functions” play a special role in the teaching and learning of mathematics at secondary school level?

During the third interview, I asked the teachers the general question “What do you think you learnt since the beginning of your research?” focusing on each research topic. All four teachers spoke about David's topic “Applications of the limit concept in mathematics and in other sciences”. Some of them came back to applications when speaking about teaching limits in schools.

As for the previous aspect of MfT limits, I defined some categories to analyse the evolution of teachers' mathematical knowledge of the social justification, as well as their ideas about how to use this aspect of limits in teaching. They are:

**Table 8-1 Categories of teachers' knowledge about the social justification**

SJ-MK1	The teacher does not acknowledge the importance of the limit concept.
SJ-MK2	The teacher knows of few applications of limits in mathematics and physics.
SJ-MK3	The teacher knows of several applications of limits in mathematics.
SJ-MK4	The teacher knows of several applications of limits in mathematics and in other sciences.

**Table 8-2 Categories of teachers' teaching ideas about the social justification**

SJ-T1	The teacher would not show the importance of the limit concept to students.
SJ-T2	The teacher is willing to explain the importance of the limit concept to students.
SJ-T3	The teacher is willing to use tasks that show the importance of the limit concept.

These categories, which also emerged from the data analysis, were used according to the following indicators.

SJ-MK1 – The teacher says that he does not understand the importance of the limit concept. This is the case of Mateus and David during the first interview.

SJ-MK2 – The teacher mentions some applications of limits in mathematics, such as areas and geometry (Abel, 1<sup>st</sup> interview) or derivatives and series (Frederico, 1<sup>st</sup> interview) and refers to applications in physics (Abel and Frederico, 1<sup>st</sup> interview).

SJ-MK3 – The teacher gives a list of applications of limits in mathematics (All teachers, 3<sup>rd</sup> interview).

SJ-MK4 – The teacher gives a list of applications of limits in mathematics and also in other sciences, such as physics, biology, chemistry (Abel and David, 3<sup>rd</sup> interview).

SJ-T1 – The teacher does not acknowledge the importance of the limit concept (SJ-MK1) and, consequently, would not show it to his students (Mateus and David, 1<sup>st</sup> interview); or the teacher knows of some applications of limits (SJ-MK2 to SJ-MK4), but does not speak of using this aspect of limits in teaching (Abel, 1<sup>st</sup> and 3<sup>rd</sup> interviews; Frederico, 1<sup>st</sup> interview).

SJ-T2 – The teacher acknowledges the importance of the limit concept (SJ-MK2 to SJ-MK4), and says that he would explain it to his students (Mateus, 3<sup>rd</sup> interview).

SJ-T3 – The teacher acknowledges the importance of the limit concept (SJ-MK2 to SJ-MK4), and says that he would give his students tasks to apply limits (David, 3<sup>rd</sup> interview).

## 8.2 David

During the first interview, when I asked David whether he knew what limits were for, he answered that he understood more or less the usefulness of limits. He explained.

Por um lado consegui chegar a ver mais ou menos o que é que, para quê é que serviam os limites, não é, porque, prontos, até um, por exemplo ao falar de limites das funções mas até um certo ponto, não é, ou até uma certa situação, as funções começam a ser monótonas, não é... Então para que o  $x$  tenda, ou para que não se calcule só... nós podíamos já tentar suprimir a monotonia e calcular, talvez mais para reduzir... (I1/D/347-66)

On the one hand I got to understand more or less what, what the limits were for, you know, even a, for example when we speak about the limit of a function up to a point, you know, or up to some situation, the functions turn into monotonic functions, you know ... Then for  $x$  to tend to, or not only to calculate ... we could try to eliminate the variation of the function and calculate, may be to reduce ...

In fact, in this quote, it is not clear what application David is speaking about. It could be about determining a limit of a monotonic function, but it does not seem to be about any application of the limit concept.

I asked him if he thought that limits were a special concept in mathematics teaching. His answer was “special, no” (I1/D/1314). I persisted with this question, and he answered:

Não vejo porque, prontos, particular, em termos de particular não estou a ver mais ou menos porquê é que... (I1/D/1320-28)

I don't think so, ok, special, in terms of special I don't see more or less why it would ...

As a conclusion, it is quite clear that, at the beginning of the research, David's knowledge about the importance of the limit concept was very poor. He was not able to mention any applications of limits and did not see limits as a special concept. Therefore I classified his mathematical knowledge as SJ-MK1 and consequently his ideas about teaching as SJ-T1 (see page 242).

David's research was about the applications of limits in mathematics and in other sciences. In his dissertation, he presented the following applications of limits in mathematics: continuity of functions, sketching graphs of functions, derivatives, integrals, series, and geometry (areas and volumes). Furthermore he presented applications in physics (instantaneous velocity, instantaneous acceleration), biology (bacteria growth), economy (average cost) and social sciences (vote probability). From these applications, he selected tasks that could be used in secondary schools and presented them in a worksheet with their solutions.

During the third interview, David confirmed that, as a secondary school student, he did not see the link between the limit concept and its applications (I3/D/82-84). He then explained that they used applications of limits in physics (instantaneous velocity).

Podia, por exemplo usamos em Física, para calcularmos a velocidade instantânea... usava mas, prontos, não pare, não pa, não passava pela cabeça de que ali estamos a aplicar os limites! (I3/D/92-101)

I could use them for example in physics, to calculate the instantaneous velocity... I would use them but, I mean, it didn't seem, it didn't, it didn't cross my mind that we were applying limits!

He discovered applications of limits to biology while doing his research (I3/D/105-114). He knew that there are some applications in chemistry but was not able to find them (I3/D/118-120).

When speaking about how he would teach limits in schools, David said that he would give students tasks involving applications of limits, not only in mathematics but also in physics, geometry, biology, economics and social sciences, using the worksheet that he produced (I3/D/2336-54). He would also try to lead other teachers to use this worksheet.

Vou tentar distribuir [a ficha] para alguns professores nas escolas, não é... e depois ver qual é que é a ideia deles. Se eles usarem, hei-de perguntar a alguns alunos, há-de ser difícil eu ir perguntar a ele se ele usou a ficha, não é! (D/I3/400-06)

I will try to give [the worksheet] to some teachers in schools, you know ... and then see what they think about it. Whether they use it, I will ask some students, it's difficult for me to ask him [the teacher] whether he used my worksheet, isn't it?

To sum up, at the beginning of the research, David did not understand the importance of the limit concept, even in mathematics (SJ-MK1; SJ-T1). As "Applications of the limit concept in mathematics and in other sciences" was the topic of his dissertation, he obviously learnt a lot about these applications and

mentioned some of them during the third interview. He made an interesting remark on how he discovered that they had used the limit concept in the study of other topics, such as instantaneous velocity, without being aware of it. I suggest that this is the result of compartmentalization of knowledge in the didactic institutions in both Mozambique and in many other countries. For example, teachers in Mozambique usually do not connect limits to derivatives or to other relevant mathematical concepts or to concepts in other topics. Furthermore David is willing to use the tasks from his worksheet in teaching limits, and suggested that other teachers also should use them.

I classified David's final mathematical knowledge as SJ-MK4 and his ideas about teaching at the end of the research process as SJ-T3 (see page 242). The evolution of his personal relation to limits in relation to the social justification of teaching this concept in schools is summarized in Table 8.5 (page 255).

### **8.3 Abel**

During the first interview, Abel spoke about the usefulness of limits in physics, without specifying which applications (I1/A/1416). In mathematics, he mentioned calculations, areas and geometry (I1/A/1424-29). When asked if he thought that limits played a special role in mathematics teaching, he mentioned again physics and geometry (I1/A/1441-45).

From these answers, we can see that Abel did not know of many applications of limits. As a mathematics and physics teacher, he knew that limits could be applied in physics and in geometry, but he did not explain how. He mentioned "calculations" as an application of limits. In fact, as said before, most of the tasks related to limits in Mozambican secondary schools are algebraic tasks. Nevertheless, calculating limits cannot be considered as an application of the limit concept. I thus classified Abel's initial mathematical knowledge as SJ-MK2 (see page 242).

Abel did not speak about showing these applications to his students. As we saw in Chapter 7, as a teacher he only taught calculations. Therefore I classified his initial ideas about teaching related to this aspect of limits as SJ-T1.



During the third interview, I asked Abel what he had learnt about applications of the limit concept during the research process. He spoke about graphical representations.

Bom, nas aplicações... dos limites na escola ... os exercícios... ... os exercícios de aplicação, digamos aquilo que se dá aos alunos... [suspira] ... Os exercícios que vou falar em termos de todo o cálculo, representação gráfica (I3/A/593-595).

Well, applications ... of limits in schools ... the tasks ... ... the application tasks, let's say what it's given to students ... [he sighs] ... I would speak of tasks in terms of calculations, graphical representation.

In this quote, Abel is speaking about the kinds of tasks given to students in schools, and not about what he learnt. In addition to calculations, which he also mentioned during the first interview, he now adds graphical representations.

When asked about the use of the limit concept, he spoke about derivatives and integrals (I3/A/705-13). He then mentioned applications in biology (bacterial growth, I3/A/724-33) and statistics (population growth, I3/A/744-45).

In physics he mentioned instantaneous velocity and mechanics (I3/A/746-53). He also stated that limits could be applied in chemistry, but was not able to mention any of these applications (I3/A/753-54). He then came back to some of these applications, trying to distinguish between the applications in mathematics and those of other sciences.

Na escola! É isso que eu disse, exercícios práticos... euh, portanto, euh, euh, representação gráfica, determinação dos limites, tudo isso. E a ligação com outras disciplinas, a Física (...) Agora cá fora vida real, euh, cá fora é isto, é que a Biologia utiliza-se, a Estatística, euh... euh, a Química, e muito mais que eu... (I3/A/758-65)

In school! That's what I said, practical tasks ... er, so, er, er, graphical representation, determination of limits, all of that. And the link to other subjects, physics (...) Now outside, real life, er, that's outside, biology uses it, statistics, er ... er, chemistry, and much more that I ...

These answers show evidence that Abel learnt a lot about applications of the limit concept in mathematics and in other subjects during the course of the research project. Comparing with the few applications that he was able to mention during the first interview, he was now able to give a list of several applications in mathematics and in other subjects. When I asked him whether he knew of these applications before, he said that he had had a very “fuzzy idea”. He also remembered having used this concept in statistics (I3/A/774-776).

When explaining how he would teach limits in schools, Abel did not mention these applications.

In conclusion, there is evidence that Abel's knowledge about applications of the limit concept evolved during the research process. During the first interview, he was only able to point out applications in physics (without specifying which applications), in mathematical calculations (without specifying which calculations), and in Geometry (determination of areas). He had probably met other applications that he did not remember, as he stated during the third interview (he had had a fuzzy idea). During the third interview he was able to mention applications in more areas and to specify some of these applications: in mathematics (graphical representations, derivatives, and integrals), in physics (instantaneous velocity and average velocity), in biology (bacterial or population growth), in statistics, and in chemistry. He became aware of the necessity of understanding the limit concept as a basis for understanding other mathematical concepts, but did not speak about using this knowledge in teaching. I then classified his final mathematical knowledge as SJ-MK4 and his ideas about teaching as SJ-T1 (see page 242).

The evolution of Abel's knowledge in relation to the social justification of teaching limits in schools is summarized in Table 8.5 (page 255).

#### 8.4 Mateus

During the first interview, Mateus stated that up to this moment he did not understand the importance of the limit concept, and seemed worried about this fact (I1/M/517-25). He only remembered using limits to define the derivative.

Sim a única utilidade que vi, como calcular o significado de derivada (I1/M/556).	Yes the only application that I was given is how to calculate the meaning of the derivative.
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In this quote, Mateus used the expression “to calculate the meaning of the derivative”. He probably wanted to speak about the definition of the derivative,

which is given as a limit, for example  $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ . The fact

that he spoke about “calculating” could reflect the fact that usually, in

Mozambican schools, the main tasks related to derivatives are to calculate them, using either the definition and therefore limits, or formulas.

He then said that he didn't remember meeting other applications (I1/M/1518-24).

These answers clearly show that Mateus's prior knowledge about the applications of the limit concept was very poor. Therefore I classified this mathematical knowledge as SJ-MK1 and, consequently, his ideas about teaching as SJ-T1 (see page 242).

He confirmed this fact during the third interview, saying that his prior knowledge about applications of limits was restricted to the definition of derivative, which he called "the definition" and not "calculating the meaning" as during the first interview. He also remembered having used limits to define some integrals, but considered limits as a tool to determine derivatives or integrals (I3/M/123-137). He reinforced this fact later on (I3/M/506-11). He also said that, during his own research, he found applications of limits in mathematics (derivative, integral, and graph sketching, I3/M/494-97). However, he seemed to consider that he learnt more about applications from David's presentations during the seminars, as he stated three times "our colleague brought in" ("o colega trouxe", I3/M/489, 501, 515) applications. David, who was researching this topic, presented applications that he had never met before (I3/M/502-3), not only in mathematics but also outside mathematics (I3/M/515-17).

Nevertheless, speaking about applications to other subjects, he did not specify which applications. He only stated that they show the importance of the concept.

Com, com aplicações dos limites, com aplicação dos limites, mostra quão importante, não só os limites, mas também a própria Matemática. Sim (I3/M/589-90).

With, with applications of limits, with application of limits, it shows how important is, not only the limits, but the mathematics itself. Yes.

From this quote, it seems that, through the importance of the limit concept, Mateus is also reflecting about the importance of mathematics.

When speaking about teaching limits in schools, Mateus said that he would show students the importance of limits in several domains (I3/M/1035-37). He also spoke about showing the importance of limits to students through the graphical interpretation of limits (I3/M/1037-40), as well as its application to calculus

(I3/M/1040-41). He would inform them that they would use limits in their further studies, for example derivatives, and integrals (I3/M/1045-47).

To sum-up, at the beginning of the research Mateus did not understand the importance of the limit concept. He did not even see the link between limits and other mathematical concepts, except for the definition of the derivative (SJ-MK1, SJ-T1). He now understands the importance of the limit concept, in particular because of the links it has with other mathematical concepts and its application to several fields. He discovered during his own research some mathematical applications and through David's presentations during the seminars other applications that he did not specify. He is willing to use these applications in teaching to show the learner the importance of this concept. I classified his final mathematical knowledge as SJ-MK3, because he did not specify during the interview which applications to other subjects he knew. In fact his knowledge could have reached SJ-MK4, but I do not have evidence of that. I classified his ideas about teaching as SJ-T2, because he said that he wanted to explain the importance of the limits to students (see page 242).

For a summary of the evolution of Mateus' personal relation, see page 255.

## 8.5 Frederico

During the first interview, Frederico stated that, when studying at the Pedagogical University, he saw that the limit concept had applications in physics (instant velocity). He then tried to explain.

Cheguei de, de ver, euh, mais ou menos, por exemplo quando se fala de, da, da velocidade instantânea (...) significa que um é um valor pequeno, então eu relaciono com, com a história de limite [o seu tema], que afinal quando se diz limite é... mais ou menos uma tentativa de encontrar, mesmo que tenhamos, seja um, uma coisa pequeninha podemos euh, conseguir entender que é limite, que, que seria limite que, da velocidade naquele estado, qual é a velocidade que é... é mais ou menos isso (I1/F/764-773).

I came to, to see, er, more or less, for example when you speak about, about, about instant velocity (...) it means that one is a small value, then I relate it to, to the history of limit [his research topic], that after all, when you say limit, it is ... more or less trying to find, even if we have, let's say a, something very small we can, er, get to understand that it is limit, that, that it would be limit that, of the velocity at this stage, what is the velocity that is ... is more or less this.

In that quote, Frederico's explanations about small values and the link between limit and velocity were not clear. He seems to be confusing the limit, which does not need to be a small value, with the interval where we calculate the average velocity and that tends to zero in order to get the instantaneous velocity. He also mentioned the history of the limit concept, trying in that way to make a link with his own research topic. Again this link was not very clear.

He also indicated derivatives as being “more related to limits” (“está mais relacionado com limites”, I1/F/781). He then tried to explain.

Aquilo está mais relacionado com limite, porque quando se fala, quando se fala de derivada significa que estamos a repartir um, o, vamos supor que estamos a repartir um certo objecto em várias partes mais, mais pequenas, então essas partes mais pequenas é que procuramos, até quando podemos conseguir repartir portanto eh, esta fracção... Hum. Então significa que aí também entra esse conceito de limite... Qual é o limite máximo que nós podemos conseguir repartir daquela maneira? (I1/F/781-800)

This is more related to limits, because when speaking about, when speaking about derivative, it means that we are dividing a, let's say that we are dividing an object in several smaller parts, then these smaller parts that we are looking for, when we are able to divide let's say, er, this fraction... Hum. Then it means that the limit concept is here again... What is the maximum limit that we can divide in that way?

Here again Frederico's explanations were not very clear. Was he speaking about derivatives or about integrals? This explanation could be interpreted as the division of an area below a curve in several small vertical areas, each of whose bases tends to zero. In that case he would be speaking about integrals and not derivatives. In any case, he seemed to have a very vague idea of the applications of limits to derivatives or integrals.

He also spoke about applications of limits to study the convergence of numerical series (I1/F/869-878). Later on, when I asked him if he thought that limits of functions play a special role in mathematics, he came back to velocity but without linking explicitly this concept with limits (I1/F/1897-1708).

In conclusion, at the beginning of the research, Frederico had some vague ideas about applications of the limit concept in mathematics. He was aware that derivatives related to limits, but was not able to clearly explain the link. He also knew that limits could be applied to study the convergence of numerical series. Regarding other applications, he knew that instantaneous velocity, and kinematics

in general, relate to limits. I would say that his knowledge of the reasons why the limit concept is taught at secondary school level was quite poor, and I classified this knowledge as SJ-MK2 and SJ-T1 (see page 242).

During the third interview, Frederico said that he now understood the importance of this concept, referring to its applications to derivatives and integrals, and to the study of functions (I3/F/42-47). He also referred to David's work, but he was not very explicit about what applications he learnt from this work.

Sobre o trabalho do, do David, do David.  
Portanto vimos também a, a aplicação do  
conceito de limite, não é... portanto em, em  
vários domínios, não é, a partir de... de  
muitos, outros conceitos que são as  
derivadas, os integrais, e, e, e outros.  
Portanto também, euh, aquilo, euh,  
facilitou, ou ajudou muito (I1/F/274-81).

About the work of, of David, of David.  
There we also saw the, the application of  
the limit concept... I mean in, in several  
areas, you know, from ... many, other  
concepts which are the derivatives,  
integrals, and, and, and others. This as well,  
er, this, er made it easier, or helped a lot.

Frederico did not specifically say that he would use these applications in teaching, but suggested that the study of limits should begin earlier, for example in Grade 10 or 11, because this concept is used for the study of other concepts (I3/F/63-64). Beginning earlier would give more time for others than algebraic activities.

To summarize, at the beginning of the research Frederico had a vague idea about applications of limits in mathematics (derivative and numerical series) and in physics (SJ-MK2, SJ-T1). Through David's presentations during the seminars, he became aware of the importance of this concept, and that it has many applications, but he did not explicitly list these applications. As with Mateus, I classified Frederico's mathematical knowledge as SJ-MK3, because he did not specify any application of limits in other sciences. This does not mean that he did not know them, but I have no evidence to classify his knowledge as SJ-MK4 (see page 242).

His ideas about showing the importance of limits in schools are not very clear. He said that he would like to begin earlier to give students other activities, but did not specify that applications of limits would be one of these activities. I then classified his ideas about teaching as SJ-T1. The evolution of his personal relation to limits in relation to the social justification of teaching limits in schools is summarized in Table 8.5 (page 255).

## 8.6 Conclusion

The evolution of the teachers' personal relation to limits regarding the social justification for teaching this concept in secondary schools can be seen from three different angles: in the first place, what they learnt about the applications of this concept in mathematics and in other subjects (mathematical knowledge), then how they analysed their prior knowledge at the end of the research process (awareness), and finally how they would use their knowledge in teaching (teaching ideas).

### 8.6.1 Evolution of teachers' mathematical knowledge

The main results from the analysis of teachers' mathematical knowledge about this aspect of limits are summarised in Table 8.3.

**Table 8-3 Evolution of teachers' mathematical knowledge about the social justification for teaching the limit concept in schools**

	Initial knowledge (1 <sup>st</sup> interview)	Final knowledge (3 <sup>rd</sup> interview, David's dissertation)
Abel	SJ-MK2	SJ-MK 4
Mateus	SJ-MK1	SJ-MK 3 (or SJ-MK 4)
David	SJ-MK 1	SJ-MK 4
Frederico	SJ-MK 2	SJ-MK 3 (or SJ-MK 4)

This table shows that all the teachers learnt about the applications of the limit concept in mathematics and in other sciences during the research process.

At the beginning of the research, all four teachers knew very little about the importance of the limit concept (SJ-MK1 or SJ-MK2). During the first interview, some of them were able to give examples of applications in mathematics (derivatives, numerical series) and in physics (instantaneous velocity) but they did not seem to fully understand the conceptual relation between these concepts. Mateus and David's following utterances expressed this fact very clearly.

Mateus: "even now I don't understand [...] the meaning of limits, the importance of limits" (I1/M/517-21).

David: “special, no” (I1/D/1314). He did not see limit as a special concept.

During the research process, they became aware of the links between the limit concept and other mathematical concepts that they studied in schools or at university (SJ-MK3). They also discovered that the limit concept had many applications in other sciences and they are now able to give some examples of these applications. This is more clearly the case for Abel and David, who listed several applications during the third interview (SJ-MK4). Mateus and Frederico were not so explicit.

This learning occurred in part from each teacher’s own research, but mainly through the presentations and discussions of David’s work during the seminars. Frederico and Mateus directly referred to David’s work as having brought in new applications. Abel did not refer explicitly to this work, but the examples that he presented during the third interview came obviously from David’s dissertation.

### **8.6.2 Awareness of their prior knowledge**

All four teachers became aware that their prior knowledge about the applications of the limit concept was very poor, as can be surmised from the following assertions made during the third interview:

- Abel said that at the beginning he had a very “vague idea” about limits’ applications.
- Mateus stated that his prior knowledge about limits’ applications was restricted to the definition of the derivative.
- David said that at school he did not see the link between the limit concept and its applications.
- Frederico said that he now understood the importance of limits.

As suggested before, this is probably the result of the compartmentalization of knowledge in many didactic institutions.

### **8.6.3 Ideas about teaching**

The results of the classification of the teachers’ ideas about teaching limits in relation to the social justification are summarised in Table 8.4 (next page).



**Table 8-4 Evolution of teachers' ideas about teaching in relation to the social justification for teaching the limit concept in schools**

	<b>Initial ideas (1<sup>st</sup> interview)</b>	<b>Final ideas (3<sup>rd</sup> interview)</b>
<b>Abel</b>	SJ-T1	SJ-T1
<b>Mateus</b>	SJ-T1	SJ-T2
<b>David</b>	SJ-T1	SJ-T3
<b>Frederico</b>	SJ-T1	SJ-T1

As shown in previous sections, all four teachers had poor mathematical understanding of the importance of the limit concept at the beginning of our work together (SJ-MK1 or SJ-MK2). As a consequence, we can surmise that they would teach this topic according to the syllabus, without linking it to other concepts, nor giving their students any tasks involving application of this concept. This is clear in the way Abel described how he taught limits (see Chapter 7).

During the third interview, only two teachers focused on the use of applications in teaching limits of functions:

- David said that he was willing to use the worksheet he produced during his research (SJ-T3). He would also like other teachers to use it.
- Mateus said that he would both show students the importance of limits in several domains, and inform them that they would use limits in their further studies, for example derivatives, and integrals (SJ-T2).

Frederico suggested that the study of limits should begin earlier, in order to give time for more activities in addition to the algebraic ones. However he did not specify tasks about applications as one of these possible activities (SJ-T1).

Abel did not speak about using applications in the study of limits (SJ-T1).

#### **8.6.4 Overview**

The evolution of teachers' personal relation to limits according to the two components, mathematical knowledge and ideas about teaching, is presented in Table 8.5 (next page).

**Table 8-5 Evolution of teachers' personal relation to limits referring to the social justification**

	Mathematical knowledge	Ideas about teaching
Abel	SJ-MK2 → SJ-MK4	SJ-T1 → SJ-T1
Mateus	SJ-MK1 → SJ-MK3	SJ-T1 → SJ-T2
David	SJ-MK1 → SJ-MK4	SJ-T1 → SJ-T3
Frederico	SJ-MK2 → SJ-MK3	SJ-T1 → SJ-T1

There is evidence that, during the research process, learning occurred regarding the reasons why the limit concept is part of the secondary school syllabus. The teachers are now able to establish links between limits and other mathematical concepts, and know that this concept can be applied in other areas of knowledge. However this does not mean that they are willing to use this knowledge in teaching.

Obviously David learnt more about this aspect of MfT limits, because it was his own research topic. In his case the learning was more active, as he had to look for applications of limits in mathematics and other sciences for his dissertation and prepared a worksheet to be used in schools. For this reason he went further in challenging the secondary school institutional relation.

The other three teachers were more passive in relation to this aspect of limits. They mainly learnt about the social justification through David's presentations at the seminars. This could explain why none of them spoke about using David's worksheet in schools. Mateus said that he would explain the importance of limits to students. As we already observed in Chapter 7 (see pages 199 and 225-26) and will corroborate in the next chapters, Mateus appeared to have a stronger mathematical background. This can explain why he was able to take into account his colleagues' findings.

To conclude, as regarding the organisation of students' first encounter with limits of functions, the new institution's relation to the limit concept shaped the teachers' personal relation to this concept in this category of *mathematics for teaching* limits, although the evolution of their personal relation was uneven, as in

the previous category, in particular concerning how they would use this knowledge in teaching.

In the next chapter, I analyse the evolution of teachers' personal relation to limits with respect to the essential features of the limit concept.

# **CHAPTER 9**

## **ESSENTIAL FEATURES OF THE LIMIT CONCEPT**

## 9 Essential Features of the Limit Concept

In the previous chapters, I have analysed the evolution of teachers' personal relation to limits according to two aspects of *mathematics for teaching* (MfT) limits: how to organise students' first encounter with the limit concept (Chapter 7), and the social justification for teaching this concept in secondary schools (Chapter 8). I concluded that for both aspects teachers' personal relation to limits had substantially evolved during the research process.

In Chapter 4, I explained that the epistemological study of limits shows that this concept can be seen from very different points of view (dynamic, static, and operational), which constitute its essential features. I also showed that in Mozambican didactic institutions the operational point of view was dominant (see Chapter 2). This chapter is dedicated to the evolution of teachers' personal relation to limits regarding the essential features of the limit concept, which is a sub-category of scholarly mathematical knowledge. It is structured as follows:

9.1. Data collection and analysis

9.2. Abel

9.3. David

9.4. Mateus

9.5. Frederico

9.6. Conclusion

### 9.1 Data collection and analysis

The essential features of a concept draw attention to the essence of this concept.

As explained in Chapter 4, there are three main features of the limit concept:

- A dynamic point of view, related to the idea of movement: when a variable  $x$  tends to a value  $a$ , the variable  $y$ , which depends on  $x$ , approaches a value  $b$ ;

- A static point of view: for  $x$  more than a determined value, the distance between the  $y$ -values and the limit are less than a certain number. There is no idea of movement; the limit is something that cannot be moved.
- An operational point of view: the limit works in accordance with rules.

The information about teachers' personal relation to the essential features of limits has been gathered through:

- The specific question asked to the teachers during all three interviews "How would you explain the limit concept to a person who doesn't know mathematics, for example a teacher of Portuguese language?" (Question EF);
- Teachers' other statements related to essentials features during the interviews.

In fact, Question EF relates to how the teachers link the limit concept to their everyday experience, while other statements about this issue usually relate to the mathematical concept. While the mathematical concept of limits can be seen from three main points of view, the everyday concept, as defined in Collins dictionary (cf. extract from Collins COBUILD Dictionary in Appendix 9.1), is usually considered in a more static way. In the everyday, a limit is mostly seen as something that cannot be transcended. This is the case of the concrete limits of a plot, city, or country, but also of more subjective concepts such as the limits of a situation, behaviour or feeling. I took this fact into account in my analysis, distinguishing, for each teacher, comments related to the everyday concept from comments related to the mathematical concept.

In order to analyse the teachers' points of view about essential features of the limit concept, their statements were classified as follows:

- Dynamic point of view – it is indicated by statements that give an idea of movement. For example, the idea of "a moving train", related to the everyday concept of limit, and expressions such as "approach", "approaching", and "you get there", associated with the mathematical concept.

- Static point of view – it is indicated by statements that give the idea of something fixed, to which we can be very close. For example, words such as “boundary”, “wall”, “place”, and “mark”, in the everyday language, and “number”, “close value” for the mathematical concept.
- Operational point of view – it is indicated by words such as “value”, “calculating”. This point of view is only applicable to the mathematical concept.

Some expressions can reflect both dynamic and static points of view. This is the case of “it approaches a fixed number” (“aproxima-se dum determinado número”) which expresses the idea of movement (it approaches), but also the idea of a static number (a fixed number). For this reason I created a new category for expressions which can be interpreted both as dynamic and static.

Speaking about “number” or “value” can represent a static or an operational point of view, according to the sentence. I classified these words as static or operational, depending on the context.

The expressions used by the teachers and the corresponding English words or expressions, for the four points of view: dynamic, static, both dynamic and static, and operational are presented in Table 9.1 (next page).<sup>18</sup>

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<sup>18</sup> Some expressions in Portuguese have several meanings in English. For example, “fronteira” which can mean “boundary” or “border”, and “balizas” that can be “goals” or “landmarks”. I used the meaning which seemed more appropriated depending on the context.

**Table 9-1 Indicators for Essential Features of the limit concept**

Point of view	Indicators (Portuguese)	Indicators (English)
<b>Dynamic point of view</b>	Aproximação Não toca até lá Chegado lá Comboio em movimento	Approach, approaching It doesn't get there Getting there Moving train
<b>Static point of view</b>	Parede Fronteira Lugar Risco Balizas Regras Regulamentos Não pode decidir (impedimento) Limitação Fim Certo valor Valor próximo	Wall Boundary/Border Place Mark Landmarks, goals Rules Regulations You can't decide (interdiction) Limitation End A particular value A very close value
<b>Dynamic and static point of view</b>	Aproxima-se dum determinado número	It approaches a fixed number
<b>Operational point of view</b>	Número Valor Cálculo de números	Number Value Calculating number

To analyse the evolution of teachers' knowledge about the essential features of limits, I defined four categories regarding their mathematical knowledge and two categories for possible use of this knowledge in teaching.

**Table 9-2 Categories of teachers' mathematical knowledge about the essential features of the limit concept**

EF-MK1	The teacher speaks about limits using mainly one of its features.
EF-MK2	The teacher speaks about limits using mainly two features, but is not aware of the different features.
EF-MK3	The teacher speaks about limits using three features, but is not aware of the different features.
EF-MK4	The teacher knows that the limit concept can be seen from different points of view.

**Table 9-3 Categories of teachers' ideas about teaching essential features of the limit concept**

EF-T1	The teacher would teach limits mainly from an operational point of view.
EF-T2	The teacher is willing to show students other features of the limit concept.



These categories were used according to the following indicators.

EF-MK1 - The teacher's utterances present only words or expressions related to one of the features (see Table 9.1). This is the case of Abel during the first interview.

EF-MK2 - The teacher's utterances present words or expressions related to two features. However, he will state afterward that he was not aware of these different features (David, 1<sup>st</sup> interview).

EF-MK3 - The teacher's utterances present words or expressions related to three features. However, he will recognise later that he was not aware of these different features (Mateus, 1<sup>st</sup> interview).

EF-MK4 - The teacher speaks about the essential features of the limit concept (Abel, Mateus and David, 3<sup>rd</sup> interview).

EF-T1 – The teacher does not acknowledge the essential features of the limit concept (EF-MK1 to EF-MK3). As a consequence he would probably teach limit from an operational point of view, according to the institutional relation (All teachers, 1<sup>st</sup> interview)

EF-T2 – The teacher acknowledges that the limit concept is not restricted to calculations (EF-MK4), and is willing to show other features to his students. He speaks about showing students “the movement” (Abel, Mateus and David, 12<sup>th</sup> seminar), “reaching the meaning of the concept” (Frederico, 12<sup>th</sup> seminar)

To classify teachers' knowledge using these categories, I first drew a table (Appendix 9.2) presenting the key words used by the teachers in their utterances related to the essential features of the limit concept during the first and the third interviews. In that table, I distinguished, on the one hand the everyday concept from the mathematical concept, and on the other hand the teachers' prior conceptions (coming from the first interview) both from their final conceptions (third interview) and from how they analyse their prior ideas afterward (also from the third interview). I then indicated, for each case, which point of view was dominant according to my classification: static, dynamic, both static and dynamic or operational.

In the teachers' quotes presented below, I indicate in bold the more significant parts of teachers' utterances, from which I drew my conclusions.

## 9.2 Abel

During the first interview, Abel spoke several times about limits as an approach (I1/A/367-71, 731, 831-32, 863), as for example:

<p>Euh, este conceito, esta <b>aproximação</b> que tende para [...] determinado, euh, valor (I1/A/367-71).</p>	<p>Er, this concept, this <b>approach</b> that tends to [...] certain, er, value.</p>
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These statements correspond to a more dynamic point of view, even if “a certain value” is more static.

When I asked him Question EF<sup>19</sup>, he was unable to answer it without using mathematical terms (neighbourhood, function  $f(x)$ ).

<p>Eu teria que começar por exemplo, o conceito de vizinhança, não é! [...] teria que dar uns certos valores, não é, valores, euh, portanto aqui é <math>x</math>, aqui <math>f</math> de <math>x</math> (I1/A/768-73).</p>	<p>I would have to start for instance, the concept of neighbourhood, you see! [...] I would have to give some values, you know, values, er, so here we have <math>x</math>, here <math>f(x)</math>.</p>
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In this quote, Abel is speaking about using a numerical register in order to find a limit. This corresponds to a more dynamic conception of limits. He then realised that he was explaining in mathematical terms and tried again.

<p>Bom. Todos <b>os valores que tendem</b> para uma determina, para um <b>determinado valor</b>... euh, mas tem um determinado limite, <b>não toca até lá</b> (I1/A/779-80).</p>	<p>Ok. All <b>the values that tend to</b> some, to <b>some value</b> ... er, but has some limit, it <b>doesn't get there</b>.</p>
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Here again both dynamic (values that tend to) and static (some value) points of view are present, but the dynamic point of view is dominant. Then Abel repeated twice “it doesn't touch” (I3/A/783,791). He seemed to conceive the mathematical concept of limit linked with the notion of asymptote of a monotonic function: the graph approaches its asymptote but does not reach it. Limits are unreachable.

Later on, when analyzing graphs, he came back to this point.

<p>E depois , euh... sim porque o, o aluno vai atribuindo os valores e vai ver [...] que <b>nem sempre chega a tocar até lá</b> (I1/A/912-</p>	<p>And then, er ... yes because the, the student will give values and will see [...] that <b>sometimes it does not get there</b>.</p>
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<sup>19</sup> Question EF: How would you explain the limit concept to a person who doesn't know mathematics, for example a teacher of Portuguese language?

916).

He now added “sometimes” to his affirmation that “it does not get there”, maybe because in one of the first figures that I presented to him the graph reaches its asymptote (Appendix 6.3, Sheet 3, Figure 2). This statement also gives an idea of movement.

When I asked him to give examples of the everyday concept, he was unable to do it and said “It’s not coming to me at this moment” (“Não me aparece tão já”, I1/A/809).

When speaking about the mathematical concept, the dynamic point of view was dominant. Therefore I classified Abel’s prior mathematical knowledge of essential features as EF-MK1 (see page 260).

During the third interview, Abel stated that for him, prior to the research, limits only meant calculations (I3/A/493-94). It seems that he became aware during the research process that he had an operational point of view about limits.

Nevertheless, during the first interview, he spoke about limits from a more dynamic point of view. This means that, even if his conception of limits was more operational, he also had the idea of limits as a dynamic process, without being aware of that. However, as he did not acknowledge the essential features of the limit concept, he was obviously not able to use them in teaching. In fact, as he said before (see Chapter 7), he mainly taught calculations. Therefore I classified his teaching ideas at the beginning of the research as EF-T1.

When I asked him Question EF (see footnote page 262), he first explained the limit concept in mathematical terms.

Bom, aqui eu teria que dar em termos matemáticos, porque trata-se de... de... bom, já entra função! [...] quando  $x$  se **aproxima de um determinado número**, não é, esse número pode ser, euh,  $a$ , não é ... euh ... é, é igual a, a um **determinado valor** que se, euh... **um determinado valor** que se encontra, portanto, digamos, euh, no eixo [...] portanto esse ... esse valor que se **aproxima** (I3/A/1792-1803).

Well, I would need to give it in mathematical terms, because it is ... it ..., well, it’s about functions! [...] when  $x$  **approaches a fixed number**, you know, this number can be, er,  $a$ , you know ... er it is, it is equal to, to a **fixed value** that is, er ... a **fixed value** located, let’s say, er, on the axis [...] I mean that ... that value which **approaches**.

In this quote, Abel showed both dynamic (approach) and static (fixed value) points of view of the mathematical concept of limits.

He then tried to give everyday examples, but faced some difficulties. He first said that “there are several examples” (“os exemplos são vários”, I3/A/1816), and then that he would ask the other person to give his/her own examples. After that he spoke about students’ conceptions.

Há alunos que entendiam que o limite é ali a **parede**! ... É a concepção que têm. Prontos, já o limite é, é, é a **fronteira**! (I3/A/1819-20)

Some students understood that the limit is here, at the **wall**! ... It’s the conception they have. Ok, the limit is, is, is the **boundary**!

It seems that Abel was reluctant to give his own interpretations of the limit concept in the everyday context. He preferred to stick to the mathematical explanation, which he learnt, or to what students would say, instead of giving examples on his own. I insisted by asking him “But you, Abel, how would you explain?” (“Mas o Abel, como é que explicaria?”, I3/A/1822)). He answered:

Ai eu teria que dizer bom o limite... pode ser... euh... **um lugar**... onde, euh, portanto, euh, **uma vez chegado lá**, não é... pronto a, aquele, a, aquele **lugar**, ou **risco**, ou limite, é o limite... Atingimos... portanto, euh... ou... se formos em termos de exemplo, poderia também pegar, euh, euh, esse exemplo de, de **parede**, por exemplo. Até ali, bom, é o limite, **não consigo ir mais além da parede**! [...] Qualquer coisa tem o seu, o seu limite, o seu **fim**! (I3/A/1824-1841)

I would have to say, well, the limit ... can be ... er ... **a place** ... where, er, I mean, er, **when you get there**, you know ... I mean to that, to, to that **place**, or **mark**, or limit, it’s the limit ... We reached ... I mean, er ... or ... in terms of examples, I could also use, er, er, this example of, of a **wall**, for example. To that point, well, it’s the limit; **I can’t go further than the wall**! [...] Everything has its own, its own limit, its **end**!

In this quote, Abel presented limits from both dynamic (you get there, I can’t go further) and static (place, mark, wall, end) points of view.

To summarise, Abel spoke about limits as a mathematical concept in terms of “approaching” a “fixed value”, which indicates a both dynamic and static point of view. When speaking about the everyday concept, the static point of view was more central (wall, boundary, place, mark) even if the dynamic point of view was also present (you get there, I can’t go further). He was able to give some examples of limits in the everyday context. Furthermore, and much more important, he seemed to be aware that before he joined the group his mathematical conception of limits was mainly operational. There is evidence that his concept image of the limit concept expanded, as well as his awareness of the different features of this concept. I classified his final knowledge about essential features as EF-MK4.

Abel taught limits at school from an operational point of view, as he explained during the 3<sup>rd</sup> interview (see Chapter 7, pages 218-220). I thus classified his initial ideas about teaching related to essential features as EF-T1. He is now aware of that and willing to “try another variant in terms of movement” (S12/1978). His experiment with a Grade 12 class, using the graphical register and a computer utility, was constructed for this purpose. I classified his ideas about teaching as EF-T2.

The evolution of Abel’s personal relation to limits regarding the essential features of this concept is summarized in Table 9.6 (page 272).

### 9.3 Mateus

During the first interview, Mateus stated that he always considered the limit of a function as a number.

Sim porque nós só, nós só vemos o **número!** [...] Mas o quê depois o número, o que vamos fazer com o número? (I1/M/1206-10)

Yes because we only, we only see a **number!** [...] But what about a number, what will we do with that number?

In this quote, the use of the word “number” seems to indicate an operational point of view. They calculated limits without understanding the meaning of the calculations.

Answering Question EF (see footnote page 262), Mateus stated.

Primeiro, ia, ia buscar a linguagem... corrente [...] a linguagem corrente, o dia a dia do próprio conceito limite. Ia buscar limite como sendo mais ou menos uma **fronteira** [...] Então a partir da, da, da **fronteira**, é como se houvesse uma **ordem**, não é, em que todos têm que chegar para aqui, só que aí nem todos podem chegar até onde estou mas são **todos ficam próximos** (I1/M/1532-49).

In the first place, I would, I would go to the everyday language [...] everyday language, day-to-day of the limit concept itself. I would pick up limit as more or less a **boundary** [...] Then starting from, from, from the **boundary**, it is like having an order [an **instruction**], I mean, everybody must get here, but not all of them can get here where I am but **all stay close to me**.

In this quote, Mateus was speaking of the everyday concept of limits, from a very static point of view (a boundary, an instruction, all stay close to me). He then explained the mathematical concept.

Então explicando a ele como limite de funções e... e eu diria a ele... **um valor que se aproxima** [...] **limite numa função é ...**

Explaining him as limit of functions and ... and I would tell him ... **a value which approaches** [...] **limit of function is ... is**

**é um valor...** é um valor que o, posso dizer um valor, não valor único mas um, **um valor aproximado** que a função pode ter independentemente dos valores do  $x$  ou pode ser quando nós tivermos **valores próximos dum certo valor  $x$** , que é dum **certo valor** do domínio... Então tem, temos **um valor muito próximo**, então este, **esse valor próximo** que nós também poderemos considerar único, então esse é que vai ser o limite que vamos considerar como sendo o limite dessa função. Não queira dizer que não haja outros valores, só que todos **vão, todos valores vão aproximar até aquele valor** que nós vamos considerar mais o menos dando este exemplo de, de **fronteira**, ou como um limite, uma **ordem** qualquer de que todos têm que fazer isto (I1/M/1553-86).

**a value ...** is a value that the, I can say a value, not single value but a, **an approximate value** that the function can take independently from the  $x$ -values or it may be when we have values **close to a certain value  $x$** , which is **some value** of the domain ... Then it, we have **a very close value**, then that, **this close value** that we also can consider unique, then this will be the limit that we will consider as the limit of this function. It doesn't mean that there is no other values, but all of them **go, all values will approach this value** that we will consider more or less giving this example of, of a **boundary**, or as a limit, some any **order** that everybody must do that.

In his answer to Question EF, Mateus faced some difficulties in explaining the concept of limits of functions. Nevertheless he presented it from both static and dynamic points of view: The static point of view was dominant (value, very close value, staying close), but the dynamic point of view was also present (values that approach). He then came back to the everyday concept, again from a static point of view (a boundary, an order).

During the interview, Mateus spoke about limits from the three points of view in a spontaneous way. I therefore classified his prior knowledge as EF-MK3 (see page 260).

At the beginning of the third interview, Mateus said that, before he joined the group, he had a static concept of limits. He saw limits as calculations, and only used a graphical representation of a limit as an interpretation.

**Antes do trabalho de investigação, antes de juntarmos o grupo, eu tinha ideia de que limites era um conceito... estático. Portanto limite só como cálculo de números.** Então é assim como... não estou a acusar mas, é assim como aprendi (I3/M/56-62).

**Before the research, before joining the group, I had an idea of limits as a ... static concept. I mean limits only as calculating numbers.** Because it's how ... I don't blame anybody but, it's how I learnt.

This quote shows clearly that Mateus became aware of his prior conception about essential features of the mathematical concept of limits during the research process. He also became aware that the limit concept can be seen from different

angles, and he classified his own point of view as static. I would rather say that he had an operational point of view (calculating a number). As for Abel, I classified his prior ideas about teaching as EF-T1 (see page 260).

Later on, when explaining how he would teach this topic, he referred to a paper by Williams addressing the issue of students' conceptions, where the example of a “**moving train**” (um “**comboio em movimento**”, I3/M/959) was given. He said that he would use this example, as well as the idea of border.

Podia também, a partir do... limite como **fronteira**, vida real, não é, o que é que nós chamamos de **fronteira**. Por exemplo, se isto fosse na, na, num **fronteira**, viver numa **zona quase fronteiriça**, então havia de, de questionar mais ou menos o que é que é **fronteira** para o aluno? Onde é que termina a **fronteira**? ... O que é que nós chamamos de **fronteira**? Então, seria a partir de, mais ou menos exemplos dessa natureza (I3/M/964-75).

I could also, starting from ... limit as a **boundary**, real life, you know, what we call **boundary**. For example, if it was at, at, at a **border**, living in an **area near the border**, then I would ask questions more or less what's a **border** for the student? Where does the **border** end? ... What do we call **border**? So, it would be starting from, more or less this kind of examples

This idea of border corresponds to a static point of view.

As during the first interview, during the third interview Mateus spoke about limits of functions from the three points of view. When speaking about the everyday concept, the static point of view was dominant, as it was at the beginning of the research, but he now added a dynamic component (a moving train), extracted from a paper about students' conceptions. The most interesting result from Mateus' interviews is that he had reflected on the essential features of the limit concept. At the end of the process he was aware that the limit concept had different features, and that he used to hold what he called a static concept, but that I would classify as an operational point of view. I classified his final mathematical knowledge as EF-MK4 (see page 260).

In terms of ideas about teaching, we can surmise that, at the beginning of the research process, Mateus would have taught limits according to the institutional relation (EF-T1). During the 12<sup>th</sup> seminar, arguing against the numerical introduction suggested by David, he said “we will stay there, limits only as numbers [...] How will students see that this number is also related to a movement?” (see Chapter 7, page 215). This shows that Mateus is willing to show

students a more dynamic conception of limits. I then classified his final ideas about teaching as EF-T2.

The evolution of Mateus personal relation to limits regarding the essential features of this concept is summarized in Table 9.6 (page 272).

## 9.4 David

Answering Question EF (see footnote page 262) during the first interview, David said that he would start from a table, a sequence and that “there was a repetition”. He seemed to face difficulties in explaining the limit concept.

Podia partir de, duma tabela, sequência não é, porque ele pelo menos terá noção do que é uma sequência, não é... de que até um certo  $\delta$ , há uma **repetição**... então para que nós não possamos estar a **repetir**... então nós tentamos delimitar a própria **repetição**, analisando... um pouco do particular depois para tentar generalizar a situação (I1/D/1343-63).

I could start with, with a table, I mean a sequence, because at least he [the student] would have an idea of what is a sequence ... that up to a certain  $\delta$ , there is a **repetition**... then to avoid this **repetition**... then we try to restrict this very **repetition**, analyzing .... a little bit, from the specific to try to generalise the situation.

The idea of a sequence could indicate a more dynamic point of view, and the word “repetition” can be seen as a more static point of view about limit, even if David’s explanations were not very clear. I then classified his prior knowledge as EF-MK2 (see page 260).

During the third interview he stated that, at the beginning of the research, he saw limits as calculations, which he called “distorted ideas”.

Eu, primeira coisa eu vinha com ideias um bocado distorcidas... distorcidas na medida em que... fiquei com, ficava com uma ideia, **estava com uma ideia de que os limites simplesmente eram cálculos. Eram cálculos em que... resolvíamos**, chegávamos a um certo resultado e depois não se fazia nada com o próprio resultado, não se fazia nenhuma interpretação do próprio resultado (I3/D/67-78).

I, first of all I came with ideas a little distorted ... distorted insofar as ... I had, I had the idea; **I had the idea that limits were only calculations. They were calculations where ... we solved** them, we reached some result and then we did nothing with this result, we did no interpretation of the result itself.

As in Mateus’ case, David became aware that he did calculations without any application of their result. For him limits meant calculations. I also classified his ideas about teaching as EF-T1.

David also answered Question EF during the third interview.



Diria que limites são... **balizas... balizas** que **podem ser alcançadas** ou podem não ser alcançadas... e que o conceito de limite de funções está ligado com o **conceito de aproximação** (I3/D/2922-30).

I would say that limits are ... **landmarks... landmarks** that **can be reached** or not ... and that the limit concept is linked with the **concept of approaching**.

In this quote, David spoke about the everyday concept of limit from a more static point of view (a landmark), with a dynamic component (they can be reached or not), and of the mathematical concept from a dynamic point of view (approaching).

As is the case with Abel and Mateus, David's understanding of limits' essential features evolved, and he could see at the end of the research that his prior understanding of limit was restricted to calculations. I also classified his final knowledge as EF-MK4 (see page 260).

During the 12<sup>th</sup> seminar, when speaking about the use of the numerical register, David argued that "it's easy to see the trend of the values" (see page 213). This shows that he was worried about showing students a more dynamic point of view of limits. I then classified his ideas about teaching related to essential features as EF-T2.

The evolution of David's personal relation to limits regarding the essential features of the concept is summarized in Table 9.6 (page 272).

## 9.5 Frederico

During the first interview, Frederico presented limits as a limitation, expressing a very static point of view.

Limite de funções é... mais ou menos o... o, **o limite de liberdade de.... de um certo cargo ou de uma certa responsabilidade** (I1/F/1652-53).

Limit of functions is ... more or less the ... the, **the limit of freedom of ... of some position or of some responsibility**.

Here Frederico started speaking about limits of functions, but in fact he meant the everyday concept, in a rather static point of view. He then went on speaking about rules and limitations.

Por exemplo na, na escola ou na aplicação de certos **regulamentos** em que há certas coisas que diz respeito a, a, a, a competência de professor mas há certas coisas que você não tem, **não pode decidir**,

For example in, in school or when following some **regulations** where there is something related to, to, to, to the teacher's competence but there are some things that you don't, **you cannot decide**, eh, eh, if

eh, eh, se não ser o director, significa que então ai você tem uma certa **limitação**, até certo ponto você só pode, euh, responder até este limite, fora disto aqui já está a transgredir a, a, **as regras** (I1/F/1660-65).

you are not the headmaster, it means that you have some **limitation**, to this point you only can, er, be responsible up to this limit, out of it you are breaking the, the, **the rules**.

Here again the everyday concept of limit is presented from a static point of view.

It means a restriction: limitation, rules, regulations, you cannot decide. At the end of the interview Frederico gave another example of limit as a boundary.

Mesmo em casa quando o, o, **o pai diz que aqui, até às tantas, quero toda a gente em casa...** Então ele [o filho] entender que o limite de eu estar fora é este! [...]E mesmo quando ele faz um, uns atropelos naquilo que está definido, podia ver que ya aquilo aqui ultrapassei limite (I1/F/1748-67).

Even at home when the, the, **the father says that here, at that time, I want everybody at home ...** Then he [the son] understands that the limit for being outside is that one! [...] And even when he has some, some failure in what has been defined, he could see that here I exceeded the limit.

In all these quotes, Frederico presented the everyday concept of limits as a very static one. He did not speak about the mathematical concept. For this reason I was not able to classify his prior knowledge about essential features of the limit concept.

During the third interview, I did not ask Frederico Question EF because we had already spent a lot of time analysing other aspects of limits and we did not have time to focus on this issue. As for the first interview, I did not classify his mathematical knowledge.

## 9.6 Conclusion

As in the previous aspect of MfT, social justification for teaching the limit concept (Chapter 8), the evolution of teachers' personal relation to the limit concept regarding its essential features can be seen from three angles: in the first place, what they learnt about the essential features of the concept (mathematical knowledge), secondly how they analysed their prior knowledge at the end of the research project (awareness), and finally how they are willing to use this knowledge in teaching (ideas about teaching).

### 9.6.1 Evolution of teachers' mathematical knowledge

The main results from the analysis of teachers' mathematical knowledge on essential features are summarised in Table 9.4. I did not include Frederico in this table because I did not have enough data to classify his knowledge.

**Table 9-4 Evolution of teachers' mathematical knowledge of the essential features of the limit concept**

	<b>Initial knowledge (1<sup>st</sup> interview)</b>	<b>Final knowledge (3<sup>rd</sup> interview)</b>
<b>Abel</b>	EF-MK1	EF-MK4
<b>Mateus</b>	EF-MK3	EF-MK4
<b>David</b>	EF-MK2	EF-MK4

The observation of this table shows that, at the beginning of the research, the ways teachers spoke about limits were uneven, ranging from only one point of view (EF-MK1) to the three points of view (EF-MK3). However, none of them was aware of these different points of view.

During the third interview, the three teachers spoke about limits from different points of view (EF-MK4).

### 9.6.2 Awareness of their prior knowledge

During the third interview, the teachers analysed their conception of limits before they joined the research group, showing that they were aware that this prior knowledge of limits was restricted to an operational point of view.

Then for the student, or even for me [as a teacher] at that time, limit was that! It was calculations! (Abel).

Before the research, before joining the group, I had an idea of limits as a ... static concept (...) I mean limits only as calculating numbers (Mateus).

I had the idea that limits were only calculations (David).

These quotes clearly show that all three teachers had reflected on their own knowledge on limits of functions, and are now able to analyse it in a critical way.

### 9.6.3 Ideas about teaching

Three of the teachers' ideas about teaching linked with the essential features of this concept are summarised in Table 9.5.

**Table 9-5 Evolution of teachers' ideas about teaching related to the essential features of the limit concept**

	Initial knowledge (1 <sup>st</sup> interview)	Final knowledge (3 <sup>rd</sup> interview)
<b>Abel</b>	EF-T1	EF-T2
<b>Mateus</b>	EF-T1	EF-T2
<b>David</b>	EF-T1	EF-T2

Data collected to analyse the teachers' knowledge of essential features do not give much information about how they are willing to use this knowledge in teaching. However, the way they explained how they would like to teach limits shows the following trends:

- At the beginning of the research process, they would probably not have showed these essential features to students, because they were not aware of them;
- At the end of the process, they would probably use several features of limits in their teaching.

### 9.6.4 Overview

The evolution of teachers' personal relation to limits according to the two components, mathematical knowledge and ideas about teaching, is presented in Table 9.6.

**Table 9-6 Evolution of teachers' personal relation to limits referring to the essential features**

	Mathematical knowledge	Ideas about teaching
<b>Abel</b>	EF-MK1 → EF-MK4	EF-T1 → EF-T2
<b>Mateus</b>	EF-MK3 → EF-MK4	EF-T1 → EF-T2
<b>David</b>	EF-MK2 → EF-MK4	EF-T1 → EF-T2
<b>Frederico</b>		EF-T1 → EF-T2

As for the two aspects of *mathematics for teaching* limits in schools presented in the previous chapters, there is evidence that this personal relation evolved in contact with the new institution. However, unlike the two first aspects, the teachers' final personal relation regarding this facet of knowledge is similar (EF-MK4 and EF-T2).

This can be explained by the fact that this aspect was not directly linked to any of the research topics, but appeared across topics during the discussions. In that way, all teachers were in the same position with respect to this aspect.

The next chapter analyses the evolution of teachers' knowledge about another aspect of mathematics for teaching limits: the use of the graphical register.

# **CHAPTER 10**

## **THE GRAPHICAL REGISTER**

## 10 The Graphical Register

In the previous chapters, I have analysed the evolution of teachers' personal relation to limits in relation to three aspects of *mathematics for teaching* (MfT) limits. For all three aspects I noted that teachers' mathematical knowledge with respect to these aspects of limits had considerably evolved and that they were generally willing to use this new knowledge in teaching. In this chapter I analyse the evolution of teachers' relation to limits regarding another aspect of this concept: the use of the graphical register.

Using graphs helps visualise and give sense to the limit concept (see pages 71-72). However, the graphical register is hardly used in Mozambican didactic institutions when studying the limit concept (see Chapter 2). As a consequence, even at university, students usually face difficulties in reading a limit from a graph or in using a graph to interpret a limit. It was to be expected, therefore, that teachers themselves were not used to graphical interpretations of limits.

A part of the first interview was dedicated to the use of the graphical register, through tasks involving limits that are unfamiliar in Mozambican institutions. The teachers' responses to these tasks were used to examine their prior knowledge of graphs.

During the third interview, I showed the teachers the same tasks that had been used for the first interview. The part of David's work dedicated to the use of limits for sketching the graph of a function was discussed during the 13<sup>th</sup> seminar, which took place after Mateus and Frederico's third interviews, but before David and Abel's. All these data are used in this chapter to look at the evolution of teachers' personal relation to limits regarding the graphical register.

This chapter is structured as follows:

10.1. Data collection and analysis

10.2. First interview

10.3. Mateus and Frederico's third interview

10.4. 13<sup>th</sup> seminar

10.5. Abel and David's third interview

10.6. Conclusion

## **10.1 Data collection and analysis**

### **10.1.1 First interview**

During the first interview I presented some tasks linking limits with graphs to the teachers (see Appendix 6.3, Sheet 3). The first three tasks involved reading limits from graphs in several different situations: when  $x$  tends to infinity (Task 1, Figures 1 to 5); when  $x$  tends to a finite value  $a$  (Task 2, Figures 1 to 4); several kinds of limits as well as domains of the represented functions (Task 3, Figures 1 to 5). Task 4 asked interviewees to sketch possible graphs of a function given two asymptotes (a vertical asymptote and a horizontal asymptote) and Task 5 to sketch graphs of functions given several limits.

The same tasks were used during the third interview in order to compare teachers' answers at the beginning and at the end of the research process. In both interviews, I did not ask the teachers to solve the tasks but presented them as possible tasks for secondary school students, asking their opinion about these tasks through questions such as:

- Do you think that these tasks could be used in secondary schools?
- Which task would be more difficult for students?
- Do you think they could help students understand better the limit concept?

By asking these questions I expected that teachers would spontaneously engage in solving the tasks or that I could suggest that they try to solve them, if I felt that they would not feel too uncomfortable. In fact, during the first interview, Mateus spontaneously engaged in solving all the tasks, and Frederico solved the tasks while answering my questions. I suggested David to solve the tasks, but I did not ask Abel to do so because he seemed to feel insecure and uncomfortable. For this reason, I do not have much information about this teacher's prior knowledge through the first interview.



### **10.1.2 13<sup>th</sup> seminar**

The 13<sup>th</sup> seminar took place in April 2004, after Frederico and Mateus had concluded and defended their dissertations, and had been interviewed for the third time. During these two teachers' third interviews, I became aware that they still had many difficulties in working in the graphical register. For this reason, I dedicated part of the 13<sup>th</sup> seminar to discussing the use of graphs in teaching limits, through a part of David's work entitled "The use of limits for sketching the graph of a function".

### **10.1.3 Third interview**

The third interview focused on three main questions:

- How the teachers analysed the evolution of their knowledge about the five research topics;
- How they would like to teach limits in schools;
- How they analysed the evolution of their knowledge about limits of functions and their ideas about its teaching during the whole process.

The graphical register was expected to appear in relation to the first question, in particular concerning Mateus' topic, different settings and registers, and in relation to the second one, the teaching of limits in schools. When speaking on how to teach limits in schools, I showed the teachers the same tasks that had been used for the first interview, and asked them whether they would use them in secondary schools. Three of them spontaneously engaged in solving the graphical tasks. I had to ask the fourth one, Frederico, to solve a task. For all teachers, a great part of the third interview was dedicated to the graphical register.

### **10.1.4 Analysing the evolution of teachers' personal relation to limits in the graphical register**

Data analysis of the four teachers' first and third interviews and of the 13<sup>th</sup> seminar helped me describe their personal relation to limits in the graphical register at the beginning and at the end of the research process. I distinguished two main aspects of their mathematical knowledge: reading limits from a graph and

sketching a graph using the limits of the function. In order to structure this analysis, I defined six categories for each aspect, designated by GRR1 to 6 (Graphical Register Reading 1 to 6, Table 10.1) and GRS1 to 6 (Graphical Register Sketching 1 to 6, Table 10.1).

**Table 10-1 Categories of teacher’s mathematical knowledge about the graphical register**

GRR1	The teacher is not able to read any limit from the graphs.
GRR2	The teacher is able to read some limits along a vertical or a horizontal asymptote (when the graph does not cross the asymptote).
GRR3	The teacher is able to read limits along a vertical or a horizontal asymptote (when the graph does not cross the asymptote), and infinite limits at infinity ( $x \rightarrow \infty$ , $y \rightarrow \infty$ ).
GRR4	The teacher is able to read limits along a vertical or a horizontal asymptote (even when the graph crosses the asymptote), and infinite limits at infinity ( $x \rightarrow \infty$ , $y \rightarrow \infty$ ).
GRR5	The teacher is able to read most limits but faces small difficulties.
GRR6	The teacher is able to read all kinds of limits.
GRS1	The teacher is not able to sketch any graph using limits or asymptotes.
GRS2	The teacher is not able to indicate any limit on axes. He is able to sketch a standard graph having two asymptotes, one vertical and one horizontal.
GRS3	The teacher indicates limits along a vertical or a horizontal asymptote as a whole branch. He does not acknowledge that drawing several branches may produce a graph that is not a function.
GRS4	The teacher indicates limits along a vertical or a horizontal asymptote as a whole branch. He acknowledges that the produced graph does not represent a function.
GRS5	The teacher indicates limits along a vertical or a horizontal asymptote as a local behaviour.
GRS6	The teacher is able to indicate any kind of limit on axes.

I defined 3 categories, GR-T1 to 3, to structure the analysis of teachers’ ideas about using graphs in teaching (Table 10.2. next page).

**Table 10-2 Categories of teacher's ideas about the use of graphs to teach limits**

GR-T1	The teacher would not use graphs when teaching limits.
GR-T2	The teacher acknowledges the importance of the graphical register in teaching limits.
GR-T3	The teacher acknowledges the importance of the graphical register and explains how he would use it or articulate it with other registers.

As for the categories defined for other aspects of MfT limits, these categories emerged from the data analysis. They were used according to the following indicators.

GRR1 – The teacher does not engage in solving the tasks. This is the case of Abel during the first interview.

GRR2 – The teacher only reads limits along a vertical asymptote (Task 2, Fig 1 and 2; Task 3 Fig 3 and 5) and along a horizontal asymptote when the graph does not cross the asymptote (Task 1, Fig 1; Task 3, Fig 2 and 5) (David, 1<sup>st</sup> interview).

GRR3 – The teacher reads limits along a vertical asymptote and along a horizontal asymptote when the graph does not cross the asymptote, as well as infinite limits at infinity (Task 1, Fig 3 and 4; Task 3, Fig. 2, 3 and 4) (David, 1<sup>st</sup> interview after my explanation, and Abel, 3<sup>rd</sup> interview).

GRR4 - The teacher reads all limits indicated before, as well as a limit along a horizontal asymptote when the graph crosses the asymptote (Task 1, Fig 5) (Frederico, 1<sup>st</sup> interview, and David, 3<sup>rd</sup> interview).

GRR4 – The teacher reads all kinds of limits but faces small difficulties (Mateus, 1<sup>st</sup> interview, and David, 3<sup>rd</sup> interview after my explanation).

GRR5 – The teacher is able to read all kind of limits without any difficulty (none of them).

GRS1 – The teacher is not able to sketch any graph in Tasks 4 and 5 (Frederico, 1<sup>st</sup> interview).

GRS2 – The teacher is only able to sketch a familiar graph in Task 4 (Abel, 1<sup>st</sup> interview).

GRS3 – The teacher indicates each limit as a whole branch in Task 4 (Mateus, 1<sup>st</sup> interview) or Task 5 (Mateus, 1<sup>st</sup> and 3<sup>rd</sup> interviews; David, 1<sup>st</sup> interview), producing a graph that does not represent a function. He does not acknowledge this fact.

GRS4 – The teacher indicates each limit as a whole branch in Task 5, producing a graph that does not represent a function. He acknowledges this fact (Frederico, 3<sup>rd</sup> interview).

GRS5 – The teacher indicates correctly each limit along the vertical and the horizontal asymptote, sometimes on the wrong side (David, 13<sup>th</sup> seminar).

GRS6 – The teacher indicates each limit along the vertical and the horizontal asymptote by a small line in Task 5 (David, 3<sup>rd</sup> interview).

GRS7 – The teacher indicates any kind of limits on axes (None of the teachers).

GR-T1 – The teacher taught limits without using the graphical register (Abel, 1<sup>st</sup> interview) or does not speak about using it in teaching (David, 1<sup>st</sup> interview).

GR-T2 – The teacher states that the graphical register should be used for teaching limits in schools but does not elaborate how to do that (Mateus and Frederico, 1<sup>st</sup> interview; Abel 3<sup>rd</sup> interview).

GR-T3 – The teacher explains how he would use the graphical register in schools articulating this register with other registers (Mateus, David and Frederico, 3<sup>rd</sup> interview)

In addition to the above, when solving the tasks, the teachers made some mathematical errors, or anticipated students' errors, related to the limit concept or to other mathematical concept. I listed these main errors as E1 to E7, as they appeared during the first interview (Table 10.3, next page).

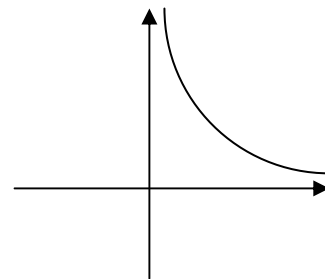
In that list, E2, E3, E4, and E7 are errors related to the limit concept. E1 relates to the concept of function, E5 to the use of Cartesian graphs, and E6 to the concept of number. The classification of each teacher's prior and final knowledge across the data analysis using these categories is presented below.

**Table 10-3 Common errors made by students when working with limits in the graphical register**

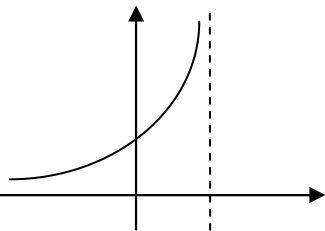
E1	Considering that a function is the analytical expression.
E2	Considering that the limit cannot be reached.
E3	Mixing-up a limit and the maximum of the function.
E4	Reading or sketching a limit as a whole branch instead of a local behaviour (see explanation in the box below).
E5	Mixing-up $x$ -values and $y$ -values, or the two axes.
E6	Not considering zero as a number.
E7	The graph cannot cross any asymptote, even a horizontal one.

**E4 - Reading or sketching a limit as a whole branch instead of a local behaviour**

Sketching a whole branch for each limit is common student behaviour. It seems that, as they are used to sketching a graph using points, they always sketch the whole branch of a graph with one stroke. When sketching graphs using limits, this strategy usually leads to bad results. For example, they would indicate the limit  $\lim_{x \rightarrow +\infty} f(x) = 0$  as in Figure A.

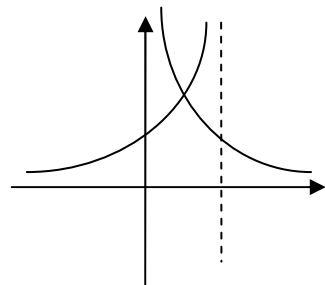


**Figure A**



**Figure B**

If another limit of the same function is  $\lim_{x \rightarrow 2^-} f(x) = +\infty$ , they would represent it as follows (Figure B).



**Figure C**

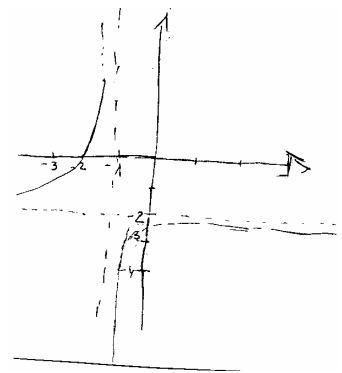
As a consequence, when combining the two limits, the resulting graph does not represent a function (Figure C).

## 10.2 First interview

### 10.2.1 Abel

Abel did not try to analyse or solve any of the tasks for reading limits from a graph. He seemed to be able to read the first limit (Task 1, Figure 1), even if he did not explicitly provide the answer. He did not seem to understand the idea behind these kinds of tasks, and his main concern during this part of the interview was trying to remember whether he used them in schools as a teacher. It seemed difficult for him to understand that a function could be given by its graph, instead of its analytical expression, as it is not usual to start from the graph in Mozambican didactic institutions. Therefore, I classified his ability to read graphs at the beginning of the research process as GRR1 (see page 277).

Abel was able to sketch a graph having both vertical asymptote and horizontal asymptotes, using his knowledge of functions usually studied in Mozambican schools (Figure 10.1). As an experienced teacher, he seemed to be strongly influenced by the secondary school institutional relation to limits. I classified his knowledge about sketching graphs as GRS2.



**Figure 10.1** Abel's graph

As explained before Abel taught limits in schools in a very algorithmic way, without linking limits with graphs (see pages 218-220). I therefore classified his initial teaching ideas as GR-T1 (see page 278). A summary of Abel's personal relation to limits with respect to the graphical register can be found in Table 10.4 (see page 284).

### 10.2.2 Mateus

Unlike Abel, Mateus spontaneously engaged in solving all the tasks for reading limits from a graph. He correctly read most of the limits. He hesitated for the limit in Task 3, Figure 2, mixing up the limit with the range, but was then able to rectify his error. He was unable to read the limit when  $x$  goes to  $-\infty$  in Task 3,

Figure 3, maybe because the vertical asymptote was the  $y$ -axis. Therefore I classified his knowledge on how to read limits as GRR5 (see page 277).

Mateus sketched a graph of a usual rational function given two asymptotes but was not able to sketch a second graph with the same asymptotes. When trying to indicate a limit, he drew a whole branch, and produced a graph that did not represent a function (Error E4, page 280). I classified his knowledge on sketching a graph as GRS3.

Considering that these kinds of tasks are not usual in Mozambican institutions, Mateus showed a relatively competent level of knowledge of reading limits from a graph. Using limits for sketching graphs seemed to be a more difficult task for him than reading limits from a graph. In addition, Mateus was able to anticipate several students' misconceptions when studying limits: the limit cannot be reached, mixing up limit and maximum.

We already saw (see page 198) that by the end of the first interview Mateus was suggesting that graphs should be used to teach limits in schools. It is difficult to say whether he had this idea before the interview. He could have been influenced by the graphical tasks presented during this interview. However he did not explain how he would use the graphs. I therefore classified his ideas about teaching as GR-T2 (see page 278).

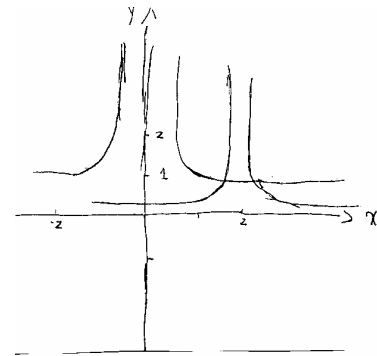
A summary of Mateus' personal relation to limits with respect to the graphical register can be found in Table 10.4 (see page 284).

### **10.2.3 David**

Unlike Mateus, David did not engage spontaneously in solving the graphical tasks, and I had to ask him to try. He correctly read a limit along a horizontal asymptote (Task 1, Fig. 1 and 2), but seemed confused when the function reaches its limit. In Task 1, Figures 3 and 5, he mixed up the limit with the maximum and was not able to read the limits at infinity. He did not know how to read a limit along a vertical asymptote (Task 2, Fig. 1). After an explanation of how to do it, he was able to read a limit of the same kind (Task 2, Fig. 2). It seems that David's knowledge on how to read limits evolved during the interview, after my

explanations. I therefore classified this knowledge as GRR2 evolving to GRR3 (GRR2  $\rightarrow$  GRR3).

Like Abel and Mateus, he was able to sketch a usual graph using two asymptotes, but unable to sketch the graph of a function given its limits. He drew a graph that did not represent a function (Figure 10.2). I therefore classified his knowledge as GRS3 (see page 277).



**Figure 10.2 David' graph**

As shown before (see page 201), during the 1<sup>st</sup> interview David did not challenge the way limits are taught in secondary schools, so I classified his ideas about teaching as GR-T1 (see page 278).

A summary of David's personal relation to limits with respect to the graphical register can be found in Table 10.4 (see page 284).

#### 10.2.4 Frederico

Frederico read most of the limits in Task 1. He did it by referring to the corresponding analytical expression of the function. It was easy for him to read limits from a graph when there was an asymptote but he faced difficulties in reading infinite limits at infinity. He was also able to anticipate some students' difficulties, and probably his own difficulties, in reading limits from a graph: mixing up the limit and the maximum of the function (E3, page 280), mixing up the two axes (E5). I therefore classified his knowledge about reading limits from a graph as GRR4 (see page 277). It is not possible from the interview to decide whether Frederico was able to sketch a graph using the limits of the function. For this reason I could not classify his knowledge in any GRS categories.

By the end of the 1<sup>st</sup> interview, Frederico suggested that it would be desirable to work more graphically in schools (see page 205). As in Mateus' case, it is not easy to know whether he had been influenced by the tasks presented during this interview. I also classified his initial knowledge as GR-T2 (see page 278).



Frederico's prior personal relation to limits with respect to the graphical register is summarized in Table 10.4 (next page).

### 10.2.5. Overview

The four teachers responded in very different ways in front of the graphical tasks during the first interview. While Abel only tried to solve Task 4, Mateus spontaneously engaged in solving most of the tasks, and I had to suggest that David and Frederico try to solve some of them. Moreover, David used my explanations to solve further tasks and it appeared that his knowledge evolved during the interview. All teachers made some error related to limits or to another mathematical concept, and some of them anticipated students' errors.

A summary of the four teachers' prior relation to limits within the graphical register using my categories (reading graphs, sketching graphs, and teaching ideas) is presented in Table 10.4, as well as the errors made and anticipated.

**Table 10-4 Teachers' personal relation to limits in the graphical register – First Interview**

	Reading graphs	Sketching graphs	Teaching ideas	Main errors made	Anticipated learner errors
<b>Abel</b>	GRR1	GRS2	GR-T1	E1	none
<b>Mateus</b>	GRR5	GRS3	GR-T2	E4	E2, E3
<b>David</b>	GRR2→GRR3	GRS3	GR-T1	E2, E3	none
<b>Frederico</b>	GRR4	n/a	GR-T2	E4	E3, E5

This table highlights the following trends.

- At the beginning of the research, the teachers' ability to read limits on graphs varied considerably, ranging from GRR1 to GRR5: Abel did not engage in reading the limits (GRR1); David was able to read some limits, and willing to learn, using my explanations to solve more tasks (GRR2 → GRR3); Frederico read most limits when  $x \rightarrow \infty$  (GRR4); Mateus read correctly almost all the limits (GRR5).
- None of them was able to use limits to sketch the graph of a function: Frederico did not even try (GRS1); Abel and Mateus drew the graph of a

rational function having two asymptotes, without really using the limits (GRS2); Mateus and David drew a graph that did not represent a function (GRS3).

- At the end of the interview, two teachers (Mateus and Frederico) suggested the use of graphs for teaching limits.
- All of the teachers made basic errors, considering that a function is its analytical expression (Abel), that the limit cannot be reached (David), mixing-up the limit with the maximum of the function (David), reading or sketching a limit as a whole branch (Mateus, David and Frederico).
- Two of the teachers, Mateus and Frederico, were able to anticipate some students' errors.

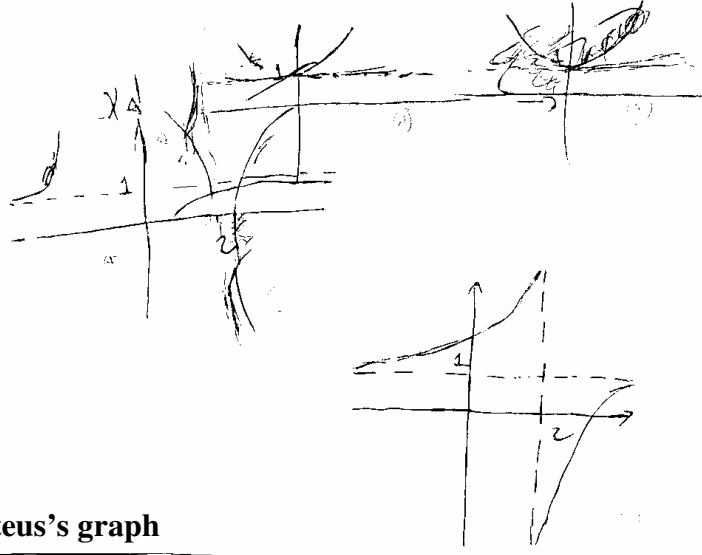
This disparity between these four teachers' personal relation to limits in the graphical register can be seen as the result of the institutional relation to graphs. None of the teachers was familiar with studying limits using the graphical register. Their different mathematical backgrounds may have helped some of them to read some limits from a graph, using their general knowledge of graphs. However, using limits for sketching a graph is a more difficult task, that none of them was able to perform.

### **10.3 Mateus and Frederico's third interview**

#### **10.3.1 Mateus**

During the third interview, Mateus showed that he acknowledged the importance of the graphical register, not only in the study of limits but also in other parts of the syllabus, such as inequalities and functions. He became aware that he did not know how to use graphs in the study of limits even during the first interview, and tried to learn more about this topic through textbooks from the library. We did not spend much time on reading limits from a graph, because he was already able to solve many of these tasks during the first interview. I classified his knowledge as GRR6 (see page 277).

We spent a lot of time solving Task 5a, which was very difficult for him. He did not seem to understand that a limit is a local behaviour of the function, and represented it as a long branch of the graph (see left graph on Figure 10.3).



**Figure 10.3 Mateus's graph**

Mateus did not seem to have a deep understanding of the function concept, in particular that to each  $x$ -value corresponds only one  $y$ -value. I classified his knowledge as GRS3 (see page 277).

Also during the 3<sup>rd</sup> interview, Mateus explained how he would teach limits in schools, beginning with everyday examples, and then going to the numerical register and introducing the graphical register (see page 224-25). He argued that “the graph has more impact” (I3/M/1020). He also said that in the first place he would ask students read limits, and then to sketch the graph of a function using its limits. This shows that Mateus reflected on how to use graphs to teach limits, and is able to articulate and argue how and why he would do it. Therefore I classified his knowledge as GR-T3.

The comparison of Mateus' personal relation to the use of graphs in the study of limits during the two interviews is presented in Table 10.5 (next page).

**Table 10-5 Evolution of Mateus’ personal relation to the graphical register**

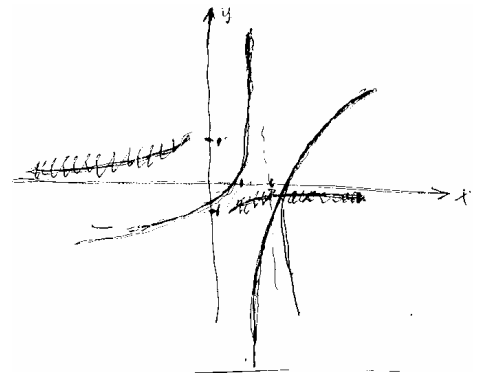
	First Interview	Third Interview
<b>Reading graphs</b>	GRR5	GRR6
<b>Sketching graphs</b>	GRS3	GRS3
<b>Teaching ideas</b>	GR-T2	GR-T3
<b>Main errors made</b>	E4	E4
<b>Main errors anticipated</b>	E2 ,E3	E6

This comparison table indicates that Mateus made some progress in reading limits from a graph, but is still unable to use limits to sketch the graph of a function. However he is willing to use graphs when teaching limits.

### 10.3.2 Frederico

When I wanted to show Frederico the graphical tasks, something unexpected happened: most of the graphs had been cleared out by my computer and I did not notice when I printed them. As a consequence it was very difficult to go on with this part of the interview and I was not able to see whether Frederico was able to read limits from the graphs or not.

I had to be very insistent for him to try to sketch some graphs using limits, probably because he was insecure of being able to do it. In fact he faced many difficulties, disconnecting the limit for  $x$  and the limit for  $y$  and drawing a whole branch for a single limit (E4, page 280). However he was able to evaluate the result and see that the graph did not represent a function (Figure 10.4). I classified his knowledge as GRS4.

**Figure 10.4 Frederico’s graph**

During the 3<sup>rd</sup> interview, Frederico said that if he had to teach limits, he would use the “graphical method” (I3/F/336-37). He explained that the observation of the graph help students understand the idea of approximation (I3/F/340-43). Later on he explained how he would introduce limits in schools, starting with geometrical tasks, and then using several registers: graphical, numerical and algebraic (see Chapter 7, pages 229-230). During the 12<sup>th</sup> seminar, he argued that he would start

with graphs. He refuted David's argument that students face difficulties with graphs asserting that the teacher should teach them to use graphs, challenging the usual mathematics teaching in schools (see pages 214-215). I thus classified his final knowledge as GR-T3.

The comparison of Frederico's personal relation to the use of graphs in the study of limits as shown during the research process (Table 10.6) indicates that he made good progress in indicating limits on a graph, but is still unable to use limits to sketch the graph of a function. However, he is willing to use graphs in teaching.

**Table 10-6 Evolution of Frederico's personal relation to the graphical register**

	<b>First Interview</b>	<b>Third Interview 12<sup>th</sup> Seminar</b>
<b>Reading graphs</b>	GRR4	GRR4
<b>Sketching graphs</b>	n/a	GRS4
<b>Teaching ideas</b>	GR-T2	GR-T3
<b>Main errors made</b>	E4	E4
<b>Main errors anticipated</b>	E3 ,E5	none

#### **10.4 13<sup>th</sup> Seminar**

During the 13<sup>th</sup> seminar (03/04/2004) we discussed the section of David's work entitled "Application of limits in sketching graphs of functions". A former version of this section had been discussed during the 10<sup>th</sup> seminar (21/02/2004). In the meantime I had interviewed Mateus and Frederico, who had already concluded their dissertations, and realised that they faced many difficulties in working with graphs. The presentation of a new version of David's work was used as a means to provoke a discussion about this register.

Abel did not participate much in the discussion during the 13<sup>th</sup> seminar, perhaps because he had not read David's work carefully, as he said himself: "I read it, er, but very quickly" ("Li, euh, mas só assim a correr", S13/149).

At the beginning of the seminar, he said that he did not understand why David was sketching the graph step by step. This intervention supports the conclusion

drawn during the first interview that he sketched the graph of a standard rational function without using the limits (GRS2, page 277).

David, whose work was being discussed, showed that he still had difficulties in reading graphs. In fact in this part of his dissertation, he did not sketch any graph by himself, but used a computer utility to sketch the graph, matched each limit with a part of the graph by erasing the remaining parts, and commented on it. When doing that, he made several errors:

- He kept a large part of the graph for each limit, considering the limit as a whole branch instead of a local behaviour (E4, page 280);
- He mixed-up the two variables in his comments (E5).
- He matched some limits with the wrong part of the graph along the vertical asymptote.

This last error, specific to David, will be explained in the next section. All these errors highlight the difficulties that David still faced with the graphical register. They also point out that the use of a graphical utility could have been an obstacle to learning.

Frederico was the most active during this seminar and detected most of David's errors, showing that he was able to correctly read all the limits along a vertical or a horizontal asymptote. As David's work did not present all kinds of limits, I classified Frederico's knowledge as GRR4, completing the information from the 3<sup>rd</sup> interview.

Mateus came late. He had detected some of David's errors, but did not have the opportunity to discuss them. For this reason it is not easy to classify his knowledge.

## **10.5 David and Abel's third interview**

### **10.5.1 David**

David's third interview took place in November 2004, more than seven months after the 13<sup>th</sup> seminar. Unlike the first interview, David engaged in solving the graphical tasks that I presented to him during the third interview. He even seemed

to enjoy these tasks and tried to solve three of the tasks for sketching graphs using the limits of the function.

In his third interview, David was able to read many more limits than at the beginning of the research process. During the first interview he only read the limits on Task 1, Figures 1 and 2, and was able to sketch a usual graph having two asymptotes. Now he was able to read most of the limits in Tasks 1 to 3. However he showed difficulties with reading the limit at infinity when the graph does not have an asymptote. As during the first interview, he learnt to read some limits during the interview through my explanations. I classified his knowledge as GRR4 evolving to GRR5 (page 277).

David was able to indicate several limits on a graph. Nevertheless, he still had some misconceptions when working with graphs: an asymptote cannot be reached, a graph cannot cross any asymptote,  $1^-$  means that the function decreases.

During the third interview, David stated that the graphical register should be explored more in secondary schools. He argued that few teachers use it due to a lack of mathematical knowledge: they do not feel comfortable with graphs, and therefore avoid them in teaching (see page 227). He then explained that he would start with the numerical register, go to algebraic calculations and then to the graphical register. He would then compare the three registers (I3/D/236-256). David showed that he reflected about how to teach limits in schools, and is able to articulate a strategy using several registers. I therefore classified his final ideas about teaching as GR-T3 (page 277).

The evolution of David's personal relation to the use of graphs in the study of limits is summarised in Table 10.7.

**Table 10-7 Evolution of David's personal relation to the graphical register**

	<b>First Interview</b>	<b>Third Interview</b>
<b>Reading graphs</b>	GRR2 → GRR3	GRR4 → GRR5
<b>Sketching graphs</b>	GRS3	GRS6
<b>Teaching ideas</b>	GR-T1	GR-T3
<b>Main errors made</b>	E2, E3	E7
<b>Main errors anticipated</b>	none	none

### 10.5.2 Abel

Abel's third interview took place in December 2004, eight months after the 13<sup>th</sup> seminar. We spent a lot of time with graphical tasks during the third interview. Unlike the first interview, Abel spontaneously engaged in solving most of these tasks. We can surmise that he felt more comfortable during this interview, or that he felt more confident about his own knowledge on limits. He did not stick to the analytical expression of the functions, as he did during the first interview, but tried to analyse the graphs directly. He was not as concerned with whether these kinds of tasks were taught in secondary schools. Working with limits through the research project probably helped Abel to release himself from the weight of the institutional relation.

Abel was able to read many of the limits that he had not attempted during the first interview. They are: limits along an asymptote when the graph does not cross the horizontal asymptote (Task 1, Fig. 1 and 2); infinite limits at infinity (Task 1, Fig. 3; Task 3, Fig. 1 and 2); limits along a vertical asymptote (Task 2, Fig. 1 and 2). However, he faced many difficulties in reading other limits, making many of the common errors students make: mixing up the limit and the maximum, mixing up  $x$ -values and  $y$ -values, considering the limit as a whole branch of the graph instead of a local behaviour. Therefore I classified his knowledge as GRR3 (see page 277).

Abel was not able to indicate a limit on a graph and said that he had never done it before. I then considered that his knowledge remained at GRS2 level.

Abel experimented with a new way of introducing limits in schools, using graphs and a computer utility. He was disappointed with the results (see pages 221), and concerned about how to apply these methods, but did not present any alternative during the third interview. Therefore I classified his ideas about teaching as GR-T2.

The comparison of Abel's knowledge of the use of graphs in the study of limits as shown during the two interviews (Table 10.8, next page) indicates an evolution in his ability to read limits from a graph. However he is still unable to sketch a graph



using the limits of the function. Nevertheless, he is willing to use graphs to teach limits.

**Table 10-8 Evolution of Abel’ personal relation to the graphical register**

	<b>First Interview</b>	<b>Third Interview</b>
<b>Reading graphs</b>	GRR1	GRR3
<b>Sketching graphs</b>	GRS2	GRS2
<b>Teaching ideas</b>	GR-T1	GR-T2
<b>Main errors made</b>	E1	E3, E4, E5
<b>Main errors anticipated</b>	none	none

## 10.6 Conclusion

This section draws conclusions firstly about the evolution of teachers’ mathematical knowledge of the use of the graphical register to study limits, then about the evolution of their ideas about how to use graphs to teach limits, and finally compares the evolution of their personal relation to limits regarding the graphical register with the evolution of other aspects of MfT presented in the previous chapters.

### 10.6.1 Evolution of teachers’ mathematical knowledge of the graphical register

Table 10.9 (next page) presents a summary of the four teachers’ mathematical knowledge of the graphical register at the beginning of the research project (1<sup>st</sup> interview) and at the end of it (3<sup>rd</sup> interview and 13<sup>th</sup> seminar), as well as the errors that they made or anticipated making.

A critical reading of this table shows the following trends.

- At the beginning of the research project, the teachers’ knowledge about reading limits from graphs varied substantially, ranging from GRR1 to GRR5.
- All of them made progress in reading limits from a graph and at the end of the process their knowledge ranged from GRR3 to GRR6.

**Table 10-9 Evolution of teachers' knowledge of the graphical register**

	<b>Initial knowledge (1<sup>st</sup> interview)</b>	<b>Final knowledge (3<sup>rd</sup> interview, 13<sup>th</sup> seminar)</b>
<b>Abel</b>	GRR1 GRS2 E: E1 A: none	GRR3 GRS2 E: E3, E4, E5 A: none
<b>Mateus</b>	GRR5 GRS3 E: E4 A: E2, E3	GRR6 GRS3 E: E4 A: E6
<b>David</b>	GRR2 → GRR3 GRS3 E: E2, E3 A: none	GRR4 → GRR5 GRS6 E: E7 A: none
<b>Frederico</b>	GRR3 n/a E: E4 A: E3, E5	GRR4 GRS4 E: E4 A: none

- The progress made by the teachers in sketching graphs using the limits of the function was not as positive, in particular for Abel and Mateus who remained at the same level (GRS2 and GRS3, respectively) and for Frederico who only reached level GRS4.
- The main difficulty faced by the teachers when trying to indicate a limit on axes was considering a limit as a whole branch of the graph instead of as a local behaviour. As a consequence they produced graphs that did not represent a function. Frederico was able to acknowledge this fact, showing that he had a deeper understanding of the concept of function.
- David is the only teacher who was able to indicate limits on axes as a local behaviour and to link these limits as the graph of the function. However he was not able to produce an unusual graph, for example one that crosses its horizontal asymptote.

The difficulties that teachers faced in using graphs to read or to interpret limits can be partly explained by the relation of Mozambican didactic institutions to limits, which does not make much room for graphs. The tasks that I presented during the interviews were very unusual for the teachers participating in the study. As a consequence they did not have a learnt strategy to solve them and could only use their general knowledge about graphs. Mateus, who appeared to have stronger

mathematical knowledge, was able to read more limits on the graphs. However, representing a limit on a graph is a more difficult task, which he was unable to perform.

The evolution of the teachers' knowledge seems to have been hindered by their poor conceptual understanding of mathematics, in particular their understanding of basic concepts such as the concept of function. All teachers produced, at some point, a graph which did not represent a function, but only Frederico was able to acknowledge this fact. They also seem to have poor conceptual understanding of an asymptote, which led them to consider that a graph cannot intersect any of its asymptotes, or to face difficulties when the asymptote is one of the axes.

David was the one who, relatively speaking, learnt the more about the graphical register during the research process. Two main inter-related reasons can be evoked to explain this.

In the first place, David's knowledge of graphs evolved during the two interviews. He always tried to solve more tasks, using my explanations when he had failed to read one of the limits. David, as the youngest of the group and a teacher with little experience, seemed to position himself more as a student than as a teacher. Abel, an experienced Grade 12 teacher, and Mateus and Frederico, experienced teachers for lower grades, expressed themselves as teachers from the first interview: Abel trying to remember whether these kinds of tasks were taught in secondary schools, Mateus and Frederico anticipating students' errors. Our conversation during the interviews was a dialogue between two teachers. The relation with David was more a teacher-student relation.

Furthermore, how to use limits to sketch a graph was part of David's dissertation. As a consequence he was in a position that required him to learn more about this activity, and also got feedback during the supervision sessions and during the 13<sup>th</sup> seminar. As a result, the way David learnt how to use graphs was from more direct teaching than for the other three teachers.

Abel was in a different position from his colleagues, as already mentioned in Chapter 7. As an experienced Grade 12 teacher, he was supposed to master the mathematical knowledge about limits of functions. As a consequence, it seems

that he was not able to position himself as a student, exposing his difficulties, and learning from less experienced colleagues, as required within the new institution. This could explain why he did not participate much in the discussion during the 13<sup>th</sup> seminar, nor did he try to sketch graphs using limits during the interviews. As a consequence his knowledge regarding the use of the graphical register for studying limits did not evolve as much as the other teachers’.

### 10.6.2 Ideas about teaching

The results of the classification of teachers’ ideas about the use of graphs to teach limits of functions are presented in Table 10.10.

**Table 10-10 Evolution of teachers’ ideas about how to use graphs to teach limits**

	<b>Initial ideas (1st interview)</b>	<b>Final ideas (3<sup>rd</sup> interview, 13<sup>th</sup> seminar)</b>
<b>Abel</b>	GR-T1	GR-T2
<b>Mateus</b>	GR-T2	GR-T3
<b>David</b>	GR-T1	GR-T3
<b>Frederico</b>	GR-T2	GR-T3

The observation of this table leads to the following comments:

- At the beginning of the research process, all four teachers only seemed to know the way limits are usually taught in Mozambican schools, which hardly includes the graphical register (GR-T1). However, at the end of the 1<sup>st</sup> interview, Mateus and Frederico suggested that the use of graphs could help students understand the limit concept (GR-T2). They could have been influenced by the graphical tasks presented during this interview.
- During the third interview, three of them (Mateus, David and Frederico) explained how they would use the graphical register in teaching, articulating this register with others, in particular the numerical and the algebraic.

- Abel experimented with a teaching sequence using graphs through a computer utility in a classroom. He was disappointed with the results and did not suggest another way of using graphs to teach limits.

The evolution of teachers' personal relation to limits according to the two components, mathematical knowledge and ideas about teaching, is presented in Table 10.11.

**Table 10-11 Evolution of teachers' personal relation to limits referring to the graphical register**

	Reading graphs	Sketching graphs	Ideas about teaching
Abel	GRR1 → GRR3	GRS2→GRS2	GR-T1→ GR-T2
Mateus	GRR5 → GRR6	GRS3→GRS3	GR-T2→ GR-T3
David	GRR2 → GRR5	GRS3→GRS6	GR-T1→ GR-T3
Frederico	GRR4 → GRR4	n/a→GRS4	GR-T2→ GR-T3

The next section compares these results with the evolution of the teachers' personal relation to limits with respect to the aspects considered in previous chapters.

### 10.6.3 Comparison with other aspects of mathematics for teaching

In previous chapters, I had analysed the evolution of teachers' personal relation to limits with respect to the way the first encounter with the limit concept could be organised in secondary schools (Chapter 7), the social justification for teaching limits in schools (Chapter 8) and the essential features of the limit concept (Chapter 9). For each of these aspects of *mathematics for teaching* (MfT) limits, I reached the conclusion that the personal relation of the four teachers was closer to the relation of the new institution at the end of the research project.

However, the evolution of the teachers' knowledge was uneven particularly with respect to the first encounter with the limit concept, and the social justification for teaching limits in schools. The differences between the teachers' personal relation to these aspects of limits at the end of the research project have been explained both by their own research topics and by how they position themselves more as teacher, such as Abel, or as student, such as David.

The evolution of teachers' mathematical knowledge of the graphical register described in this chapter shows that, at the end of the research process, none of the teachers was able to flexibly read all kinds of limits on a graph. David was the only one able to indicate limits on axes as a local behaviour. Even Mateus, whose initial knowledge was the strongest (GRR5 and GRS3), did not seem to have made much progress in working with graphs (GRR6 and GRS3). However all of them became aware that using the graphical register in the study of limits would help students understand this concept better (GR-T2 and GR-T3).

I already suggested that teachers' learning about the graphical register had been hindered by their poor conceptual understanding of mathematics. Basic concepts such as the concepts of number, function, asymptote and limit itself do not seem to be fully conceptualised. The lack of clarity of these concepts for these teachers appeared through the main errors made when solving the graphical tasks.

E6 (not considering zero as a number), which appeared when reading a limit along one of the axes, shows that the concept of number is not fully acquired.

E1 (considering a function as its analytical expression), sketching a graph that does not represent a function as a consequence of E4 (the limit as a whole branch instead of a local behaviour) and E5 (mixing-up the two axes) indicate a poor understanding of the function concept.

E2 (the limit cannot be reached) and E7 (the graph cannot cross any asymptote) show a poor understanding of the concept of asymptote.

E2, E3 (mixing-up the limit and the maximum of the function), and E4 indicate a poor understanding of the limit concept.

Reading limits from a graph or sketching a graph using the limits of the function requires a deep understanding of basic mathematical concepts. Knowing several ways of organising students' first encounter with the limit concept, knowing why the limit concept is taught in secondary schools, or knowing that limits has different features each involves a more general mathematical knowledge. This could explain why the teachers were able to learn more about the first three aspects of MfT than about the graphical register. It seems that when a deep understanding of mathematical concept is lacking, the institution of research is not

sufficient to overcome the difficulties. David, who benefited from a more direct teaching about the use of graphs during the supervision sessions, is the only one able to indicate a limit as local behaviour at the end of the process. This suggests that in such a case, more direct teaching, including explanations and solution of tasks should take place. This is what happened to David through his research of applications of limits to sketch the graph of a function and discussions during the supervision sessions.

These results point out a limitation of teachers' learning through the research group: the learning through discussions within a group is not effective when teachers do not have a deep understanding of the concepts involved.

However, as with previous aspects of MfT limits, the way the teachers positioned themselves during the seminars and during the interviews seem to have influenced their learning of limits within the graphical register. Abel, who positioned himself as an experienced Grade 12 teacher, did not even try to solve the tasks for sketching graphs using the limits of the function during the interviews, and did not participate in the discussion about David's work during the 13<sup>th</sup> seminar. As a result, he is the one whose knowledge about the graphical register was more limited at the end of the process. David, who positioned himself more as a student than as a teacher, learnt from his colleagues during the seminars and improved his knowledge of the graphical register during the 3<sup>rd</sup> interview, using my explanations for solving more tasks.

As already mentioned, this points out a limitation of the new institution, the institution of research, where the teachers were required to learn from each other. A learner-teacher, who is becoming a teacher, even if he has some teaching experience, has to learn mathematics and therefore positions himself as a learner. A teacher-learner, who is an experienced teacher, even if he is concluding his training, positions himself as an expert in the knowledge. As a consequence he is less likely to be prepared to challenge his knowledge.

This contention will be supported by the analysis of the evolution of teachers' personal relation to the  $\epsilon$ - $\delta$  definition, which also requires a deeper conceptual understanding of mathematics, presented in the next chapter.

# CHAPTER 11

## THE $\varepsilon$ - $\delta$ DEFINITION



## 11 The $\varepsilon$ - $\delta$ Definition

In Chapter 2, I analysed the Mozambican Secondary School institutional relation to limits using the notion of mathematical organisation (MO) as developed by Chevallard (1999). Following Barbé et al. (2005), I considered that the reference mathematical organisation was structured into two regional MOs: MO<sub>1</sub>, the algebra of limits, and MO<sub>2</sub>, the existence of limits, based on the  $\varepsilon$ - $\delta$  definition. I showed that some trace of MO<sub>2</sub> could be found in the syllabus and in one of the worksheets elaborated by teachers, but was completely absent in the examinations and in the other worksheet. I also showed that at the Pedagogical University the same dichotomy existed between the merely algebraic practical block belonging to MO<sub>1</sub> and the knowledge block belonging to MO<sub>2</sub>, not used in practice.

In Chapter 4, I argued that, given the difficulties inherent in the formal definition, tasks in the formal setting were not suitable for secondary school level. Nevertheless, it was important for mathematics teachers to understand them (see page 111). This chapter analyses the teachers' personal relation to the  $\varepsilon$ - $\delta$  definition and its evolution through the research process.

It then provides a summary of the main results of data analysis according to the five aspects chosen for this study: the first encounter, the social justification, essential features, the graphical register, and the  $\varepsilon$ - $\delta$  definition.

This chapter is structured as follows:

11.1. Data collection and analysis

11.2. Abel

11.3. Mateus

11.4. David

11.5. Frederico

11.6. Conclusion

11.7 Overview of data analysis

## 11.1 Data collection and analysis

Data from the three teachers' interviews and the third seminar were used to analyse the evolution of teachers' personal relation to the  $\varepsilon$ - $\delta$  definition.

During the first interview, I asked the teachers the following questions related to the  $\varepsilon$ - $\delta$  definition:

- Did you understand the  $\varepsilon$ - $\delta$  definition? When?
- Which definition should be taught in secondary schools [Showing Appendix 6.3, Sheet 1]? At what level do you think that a formal definition should be taught? Why?

When the third seminar, dedicated to discussing the  $\varepsilon$ - $\delta$  definition, took place in April 2004 (see Timetable in Appendix 6.4), all teachers were still at an early stage in their research. For this reason I considered teachers' utterances both during this seminar and the first interview as indicative of their prior personal relation to the  $\varepsilon$ - $\delta$  definition. A detailed analysis of this seminar can be found in Appendix 11.1.

During the second interview, which took place just after the third seminar, all teachers spontaneously spoke about this definition. We did not discuss the definition any more during the seminars, because it hardly had a direct link with any of the research topics. Therefore I considered teachers' utterances during the second interview as indicative of their final personal relation to the definition, together with the third interview.

As for the previous aspects of MfT limits, I defined categories, emerging from the data analysis, to classify teachers' mathematical knowledge of the  $\varepsilon$ - $\delta$  definition, D-MK1 to D-MK1 (see Table 11.1 next page) and their ideas about teaching this definition in secondary schools, D-T1 to D-T3 (see Table 11.2 next page).

**Table 11-1 Categories of teachers' mathematical knowledge about the  $\varepsilon$ - $\delta$  definition**

D-MK1	The teacher is not able to correctly write the $\varepsilon$ - $\delta$ definition.
D-MK2	The teacher correctly writes the $\varepsilon$ - $\delta$ definition, but is not sure about it.
D-MK3	The teacher is sure about the correct $\varepsilon$ - $\delta$ definition, but is not able to explain it.
D-MK4	The teacher is sure about the correct $\varepsilon$ - $\delta$ definition, and is able to explain it.

**Table 11-2 Categories of teachers' ideas about teaching the  $\varepsilon$ - $\delta$  definition**

D-T1	The teacher would teach the $\varepsilon$ - $\delta$ definition at secondary school
D-T2	The teacher acknowledges students' difficulties in understanding the definition. As a consequence he would teach the $\varepsilon$ - $\delta$ definition without applications.
D-T3	The teacher acknowledges students' difficulties in understanding the definition. He is inclined not to teach it but is not sure about that.
D-T4	The teacher acknowledges students' difficulties in understanding the definition and, as a consequence, would not teach the $\varepsilon$ - $\delta$ definition.

These categories were used according to the following indicators.

D-MK1 – The teacher is not able to write the correct definition. This is the case of Abel during the 3<sup>rd</sup> seminar.

D-MK2 – The teacher hesitates in writing the correct definition, or writes it correctly but, when challenged by a comment about it, he changes this definition (David, 2<sup>nd</sup> interview).

D-MK3 – The teacher writes the correct definition and stands up for this definition (Mateus, 3<sup>rd</sup> seminar).

D-MK4 – The teacher writes the correct definition, stands up for this definition, and is able to explain it (none of the teachers).

D-T1 – The teacher taught the  $\varepsilon$ - $\delta$  definition (Abel, 1<sup>st</sup> interview).

D-T2 – The teacher says that he would teach the  $\varepsilon$ - $\delta$  definition, but without asking students to apply it, because he is aware of the difficulties that they face (Mateus, David and Frederico, 3<sup>rd</sup> interview).

D-T3 –The teacher says that he acknowledges students’ difficulties in understanding the definition. However he does not know whether to teach it or not (Abel, 3<sup>rd</sup> interview).

D-T4 - The teacher says that he would not teach the  $\varepsilon$ - $\delta$  definition, because he is aware of the difficulties that students face (David and Frederico, 1<sup>st</sup> interview).

It was difficult to classify teachers’ knowledge of the  $\varepsilon$ - $\delta$  definition using these categories for the following reasons:

- I did not collect systematic data about their knowledge of the definition. For example, during the first interview, I did not ask them to write it down but showed them this definition within a list of four definitions.
- The discussion during the third seminar helped me classify the knowledge of some teachers (Abel, Mateus), but others did not participate much in the discussion (David and Frederico).
- During the third interview, we spent a lot of time with graphical tasks and discussion about other aspects of *mathematics for teaching* limits and, as a consequence, we did not dedicate much time to the definition.

The limitations of the classification will be explained when necessary.

## 11.2 Abel

During the first interview, Abel said that he studied the  $\varepsilon$ - $\delta$  definition in Germany. He had many difficulties with it and had to ask some colleagues to explain it (I1/A/421-491). He seemed very insecure when he stated: “Well I think that I understood, in my own way, you know” (“Bom, acho que entendi, a minha maneira, não é”, I1/A/489). As a teacher he taught the  $\varepsilon$ - $\delta$  definition and observed that students did not understand it but that they memorised it and ‘sang’ it. He stated that the 2<sup>nd</sup> definition (Sheet 1) would be more appropriate for secondary school students (I1/A/691-758).

At the beginning of the third seminar, Abel positioned himself as an experienced teacher, willing to explain the  $\varepsilon$ - $\delta$  definition to his colleagues. He spontaneously went to the blackboard, wrote the analytical expression of a function

$f(x) = \frac{(2x+1)(x-1)}{(x-1)}$  and explained that, in the numerical register, we can

choose values for  $x$  in a neighbourhood of 1, which go to one from the left or from the right. He drew a graph, wrote  $1-\delta$  and  $1+\delta$  on the  $x$ -axis, and  $\varepsilon+1$  and  $\varepsilon-1$  on the  $y$ -axis, which he changed for  $\varepsilon+3$  and  $\varepsilon-3$  (S3/47-64). This shows that Abel was not really sure about the graphical interpretation of this limit. He correctly indicated the interval  $]1-\delta, 1+\delta[$  on the  $x$ -axis, but indicated a wrong interval on the  $y$ -axis: on the first place  $]\varepsilon-1, \varepsilon+1[$ , and then  $]\varepsilon-3, \varepsilon+3[$ , both intervals centred in  $\varepsilon$  instead of 3. I then told him that it should be  $3-\varepsilon$  and  $3+\varepsilon$  and he changed again (S3/71).

He wrote the following definition on the blackboard for the specific

limit  $\lim_{x \rightarrow 1} f(x) = 3$ :

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : |1-a| < \delta \Rightarrow |f(x)-a| < \varepsilon$$

[1-  $a$  instead of  $x-1$  in the first modulus,  $a$  instead of  $f(a)$  in the second modulus]

A discussion arose about this definition, and he changed it for

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : |x-1| < \delta \Rightarrow |f(x)-3| < \varepsilon$$

which is the correct definition for this specific limit.

A discussion arose about the roles of  $\varepsilon$  and  $\delta$ . During this discussion, Abel wrongly stated that the definition starts with  $\delta$  instead of  $\varepsilon$ . He consequently wrote a new definition changing the roles of  $\varepsilon$  and  $\delta$ :

$$\forall \delta > 0 \quad \exists \varepsilon > 0 : |x-1| < \delta \Rightarrow |f(x)-3| < \varepsilon.$$

This shows that Abel's knowledge of the definition was not stable. He probably memorized it but without a profound understanding, as he will confirm during the second and the third interviews. He was also unable to give a clear graphical interpretation of this limit.

Later on, he also faced difficulties when working in the numerical register: he was unable to choose an arbitrarily small  $\varepsilon$ -value, and to solve the inequality  $|x - 1| < 0.1$ . When solving this inequality he wrote on the blackboard

$$\begin{aligned}\vee |x - 1| &< -0,1 \\ \vee x - 1 &< -0,1 \\ x &> 0,9\end{aligned}$$

This solution presents several errors:

- He wrote  $|x - 1| < -0.1$ , which does not make sense in this context.
- He incorrectly used the symbol  $\vee$ .
- He wrongly changed the inequality symbol  $<$  for  $>$ .

It seems that Abel was trying to solve the inequality using his memory instead of his understanding of the solution. He told me during the 2<sup>nd</sup> interview that he was embarrassed by the errors that he made during the seminar (I2/A/232-56).

However he exposed himself by going to the blackboard to explain the definition and solve the tasks.

As a conclusion, we can say that Abel memorized the definition but did not understand it. He was not able to write it without his colleagues' support.

Furthermore, he showed a weak mathematical knowledge of basic concepts.

Therefore I classified his prior mathematical knowledge of the definition as D-MK1 (see page 301). As a teacher, he taught the  $\varepsilon$ - $\delta$  definition, although he was aware that students did not understand it. I then classified his prior ideas about teaching the definition as D-T1.

During the second interview, Abel said that he realized during the seminar that he didn't understand the definition (I2/A/57-69). Later on, he correctly wrote the definition, and asked me what happened to the definition when the limit does not exist (I2/A/216-31). I explained that the definition was useful to demonstrate that a function had a certain limit, but did not help determine this limit (I2/A/260-484). He then explained that in schools he taught the definition just as it was in textbooks.

E depois, nessa definição... se dava e às vezes até posso dizer com caderno ali, dava a definição, prontos está aqui definição tal como está no livro, acabou! Mas nunca exigi no exame, no, no ponto, que dêem-me a definição! Nada! ... Mas também eu questiono, mesmo para esses alunos agora, acho que não há necessidade de se exigir isso, não sei! Porque ensinei eu, entendo agora! (A2/A/959-67)

And then, this definition ... we taught it and sometimes I confess, looking at the exercise-book, I gave the definition; ok here is the definition as in the book, that's it! But I never asked the students to give me the definition in the exams, in, in the test! No! ... But now I ask myself, even these students now, I think that they do not need it, I don't know! Because I who taught it, I just understood now!

From that quote we can see that this issue was challenging for Abel. He was questioning his own teaching of the definition transcribed from a book, and also the institutional relation: is it worth teaching this definition in secondary schools? Nevertheless his challenge of the institutional relation was restricted.

Cá por mi eu digo que, a, acho que não, não, não, não haveria necessidade de se cingir tanto ou exigir-se tanto do aluno que, que memorize esta definição. Quer dizer, pode-se dar assim numa forma formal que, de que bom, nós já falamos isso de limite mas então em termos simbólicos, o que é que é isso? Ah! Pode-se dar mas, que o aluno seja, não sei, não sei (I2/A/1001-5).

For me I say that, I think that we don't need to keep that much or to demand that much from the student to memorize the definition. I mean, we could give it in a formal way that, well, we already spoke about limits but in symbolical way, how would it be? Ah! We can do that but the student had to be, I don't know, I don't know.

From that quote we can see that Abel acknowledged that it is not worth asking secondary school students to memorize a definition that they do not understand. However he was not able to challenge the institutional relation as far as to suggest not teaching the definition at all.

During the third interview, Abel repeated that, at the beginning of the research, he thought that he understood the  $\varepsilon$ - $\delta$  definition and realized that he did not during the 3<sup>rd</sup> seminar (I3/A/64-76). When I asked him whether he felt that he understood the definition or not, now, he answered:

É isso que eu digo, bom a pessoa, numa fase à outra, ali entendi, agora entendo, mas eu penso que entendi (I3/A/942-48).

This is the point, well a person, from one stage to another, there I understood; now I understand, but I think that I understood.

From that quote, it seems that Abel was not sure whether he understood the definition or not. For this reason I classified his final knowledge as D-MK3 (see page 301).

Abel also expressed doubts about teaching the definition in schools, because he read papers defending different opinions.

Quer dizer, a dúvida aparece-me quando há, há correntes... euh, que... uns artigos é preciso dar-se a definição porque se ensina Matemática, é preciso, ou, outras correntes, mas eu cá por mi digo que não (I3/A/917-22).

I mean the doubt appears when there are trends ... er, that ... in some papers the definition should be taught because we are teaching maths, we need it, or, other trends, but I would say that no.

From that quote, we can see that Abel seemed very unconfident about what to do but more inclined not to teach the definition. Therefore I classified his ideas about teaching as D-T3.

The evolution of Abel's personal relation to the  $\varepsilon$ - $\delta$  definition is summarized in Table 11.5 (page 315).

### 11.3 Mateus

During the first interview, Mateus explained that he studied the  $\varepsilon$ - $\delta$  definition at the Nautical School and then at PU (I1/C/279-310). At that time he did not understand this definition. As a consequence, he just memorized it (I1/C/330-347). He said that in secondary schools he would teach in the first place the 1<sup>st</sup> definition, and then the 2<sup>nd</sup> one (Sheet 1). He then said that it would be difficult for students to understand the Greek letters, which he called deltas and gammas. However he did not say whether he would teach the  $\varepsilon$ - $\delta$  definition (I1/C/566-666). I therefore classified his ideas about teaching the  $\varepsilon$ - $\delta$  definition as D-T3 (page 301).

Mateus participated quite a lot in the discussion during the third seminar (see Appendix 11.1). At the beginning of the seminar, he asked for the meaning of the symbols  $\varepsilon$  and  $\delta$ , showing that he was not afraid of presenting his own difficulties. During the discussion on whether the definition starts with  $\varepsilon$  or  $\delta$ , he insisted that we have to start with a given value of  $\varepsilon$  (arbitrarily small), showing that he was sure of this knowledge. He immediately understood the task when I suggested a shift to the numerical register, and gave orientations for Abel to solve the task on the blackboard. He then gave him instructions to solve the inequality  $|x - 1| < 0.1$ . He was also the first one to answer when I asked for the limit of the function when  $x$  tends to one.



To sum up, at the beginning of the seminar Mateus knew the  $\varepsilon$ - $\delta$  definition, but was not able to interpret it. He was aware of his difficulties and freely presented them to his colleagues. He also showed that he had a good general mathematical knowledge. I classified his prior knowledge of the  $\varepsilon$ - $\delta$  definition as D-MK3 (see page 301).

During the second interview, Mateus analysed what happened during the third seminar. He explained some of the difficulties that came out: working with Greek letters, the transition between the limit as an approach and the formal definition, the role of  $\varepsilon$  and  $\delta$ . He said that he now understood the definition (I2/C/151-257).

During the third interview, he repeated that at the beginning of the research process he did not understand the formal definition (I3/C/74-123). He did not speak about his actual understanding. He said that, as a teacher, he would give the students the formal definition after two other definitions: the first one, that he called the “intuitive” definition and the second one (Sheet 1). He added:

Então a terceira [ $\varepsilon$ - $\delta$ ] definição para mim seria uma definição já final, depois do próprio aluno entender o significado do, de limite e o próprio aluno pelo menos dando-me por suas próprias palavras do que entende por limite (I3/C/1151-54).

Then the third [ $\varepsilon$ - $\delta$ ] definition, for me, would be a final definition, after the student understood the meaning of, of limit and the student himself explained with his own words what limit means for him.

He would give it without any further application (I3/C/1167-68). I thus classified his ideas about teaching as D-T2.

I do not have much information about Mateus’ final knowledge of the definition. As said before (see page 302) I did not collect systematic data about the definition. Mateus seemed quite sure that he understood it, but I do not have evidence that he was right. I classified his knowledge as D-MK3/4.

The evolution of Mateus’s personal relation to the  $\varepsilon$ - $\delta$  definition is summarized in Table 11.5 (page 315)

## 11.4 David

During the first interview, David said that at secondary school they had to memorise the  $\varepsilon$ - $\delta$  definition and calculate limits using it (I1/D/159-246). This statement points out his weak understanding of this definition, which in fact does

not allow determining limits. He faced quite the same situation at the Pedagogical University. He stated that he still did not understand this definition (I1/D/319-338). In his opinion the 1<sup>st</sup> and the 2<sup>nd</sup> definitions (Sheet 1) should be taught in secondary schools (I1/D/456-82). Speaking about the  $\varepsilon$ - $\delta$  definition, he said:

Agora, falar da terceira é um bocado complexo porque pronto quando, quando começamos a envolver várias siglas, várias letras, aí o estudante já fica um bocado limitado, fica um bocado confuso. Primeiro precisa saber o que é esta história de  $\varepsilon$ , o que é o  $\delta$ , o que é, então assim ... (I1/D/486-95).

Now, to speak about the third one is a little complex because, you know, when, when we began involving symbols, several letters, then the student feels limited, he feels a little confused. First he needs to know what do we mean with  $\varepsilon$ , what this  $\delta$  is, what it is, so in that way ...

He then stated that it should be eliminated (I1/D/504). I therefore classified his ideas about teaching the  $\varepsilon$ - $\delta$  definition as D-T4 (see page 301).

David did not participate much in the discussion during the seminar. However, his statements were always right (see Appendix 11.1). We can surmise that, as the youngest of the group and confronted with experienced teachers such as Abel, Ernesto and Frederico, he did not feel comfortable to give his own ideas. I classified his prior knowledge as D-MK1.

During the second interview, David said that he enjoyed the third seminar discussion because it helped him understand better the  $\varepsilon$ - $\delta$  definition, in spite of his fear of  $\varepsilon$  and  $\delta$  (I1/D/315-58). The graphical representation had been a good support for him to understand (I2/D/362-431).

When I steered the conversation to the  $\varepsilon$ - $\delta$  definition during the third interview, he immediately tried to write it down. He wrote two wrong definitions before getting the right one (I3/D/1821-1856). I then asked him whether he understood the definition or just memorised it. He said that he wrote it by heart (I3/D/1858-1903). He also stated that this definition did not help students understand the limit concept (I3/1917-40). For this reason, even if you learn it, you will forget it (I3/D/1949-54). Nevertheless, later on during the interview, he said that he would teach it in school because any concept must have a definition, even if the students do not understand it

A definição eu havia de dar! Porque é um conceito, e todo o conceito tem que ser

I would teach the definition! Because it is a concept, and any concept must be defined

definido... mas, eu particularmente acho ... but I myself think that I would not use it  
que haveria de usar muito pouco (I3/D/74- much.  
79).

Considering his hesitations in writing down the correct definition, I classified his final knowledge as D-MK2. I classified his ideas about teaching as D-T2 because he said that he would teach the definition but not use it.

For a summary of the evolution of David's personal relation to the  $\varepsilon$ - $\delta$  definition, see Table 11.5 (page 315).

## 11.5 Frederico

Frederico did not remember studying the  $\varepsilon$ - $\delta$  definition at the Faculty of Education (I1/F/223-241). He learnt it at PU, but did not understand the meaning of the Greek letters (I1/F/605-14). Up to now, he did not understand it (I1/F/722-28). Nevertheless he was interested in understanding this definition and it is one of the reasons why he chose to join the research group (I1/F/732-757). He also stated that in secondary schools it would be better to teach the 1<sup>st</sup> and the 2<sup>nd</sup> definitions (Sheet 1) and not the  $\varepsilon$ - $\delta$  one (I1/F/944-967). I therefore classified his ideas about teaching the  $\varepsilon$ - $\delta$  definition in schools as D-T4 (see page 301).

At the third seminar, the only intervention by Frederico was during the discussion about starting with  $\varepsilon$  or  $\delta$ . This intervention showed that he did not have a good knowledge of the definition (see Appendix 11.1). I then classified his prior knowledge as D-MK1.

During the second interview, Frederico said that the third seminar allowed him understand the definition (I2/F/722-33). He then tried to explain the role of  $\varepsilon$  and  $\delta$ .

Então fiquei claro que, afinal de conta não é o $\delta$ , sempre nós atribuímos um valor a $\varepsilon$ e procuramos qual é o valor que faz com que este valor de $\varepsilon$ aproxima a um determinado valor, portanto quando o $\delta$ aproxima também a um valor fixo (I2/F/37-49).	Then it became clear that, after all, it's not $\delta$ , we always choose an $\varepsilon$ -value and look for the value which allows the $\varepsilon$ -value to approach a certain value, I mean when $\delta$ also approaches a fixed value.
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Frederico faced difficulties in explaining the meaning of  $\varepsilon$  and  $\delta$  and the relation between their values, but he seemed to understand that the choice of an  $\varepsilon$ -value (arbitrarily small) determines the  $\delta$ -value.

During the third interview he repeated that, before the third seminar, he was confusing the roles of  $\varepsilon$  and  $\delta$ .

Aí, eu fiquei de facto claro que aquela ideia que nós tínhamos, de hábito de que sempre as variáveis ficam, no, no eixo das abcissas, e, euh, euh, no eixo das ordenadas sempre é, é o valor da função. E nós, euh, e nós, e outros, então fazia o contrário que é delta para podermos ter, euh, o epslom. Mas aí já fiquei claro que não, aí sempre é dado o epslom [arbitrariamente pequeno], então para encontrarmos delta (I3/F/553-72).

There, it became clear that this idea we had, the habit that the variables always stand on, on the  $x$ -axis, and, er, er, on the  $y$ -axis always is, is the function values. And we, er, and we, and others as well, then we did the opposite which is delta in order to get, er, the epsilon. But there it became clear that no, the epsilon value is given [arbitrarily small] in order to find the delta value.

From that quote, we can see that Frederico understood that he was wrong, but also the reason why learners usually make this error: the roles between dependent and independent variables are inverted.

Even so, during this interview, he confused the definition of a limit with the definition of the derivative (I2/F/911-993). As his colleagues had done, he said that he would teach the  $\varepsilon$ - $\delta$  definition, but without any application (I2/F/997-1047). I also classified his ideas about teaching the definition as D-T2.

As for Mateus, it is difficult to classify Frederico's final knowledge of the definition, because I did not ask him to write it down. Considering that he seemed to understand that  $\delta$  depends on  $\varepsilon$ , but mixed up this definition with the definition of a derivative, I classified his knowledge as D-MK2.

The evolution of Frederico's personal relation to the  $\varepsilon$ - $\delta$  definition is summarized in Table 11.5 (page 315).

## 11.6 Conclusion

The analysis of teachers' comments related to the  $\varepsilon$ - $\delta$  definition leads to several conclusions. In the first place, it shows the evolution of their mathematical knowledge of the definition through the research process. Secondly, it points out some of the difficulties that learners face in understanding the definition. Finally it also shows teachers' point of view about the teaching of this definition in secondary schools. These issues are addressed separately.

### 11.6.1 Evolution of teachers' mathematical knowledge of the $\varepsilon$ - $\delta$ definition

The evolution of teachers' mathematical knowledge of the  $\varepsilon$ - $\delta$  definition according to my categories is summarized in Table 11.3.

**Table 11-3 Evolution of teachers' mathematical knowledge of the  $\varepsilon$ - $\delta$  definition**

	<b>Initial knowledge</b> (1 <sup>st</sup> interview, 3 <sup>rd</sup> seminar)	<b>Final knowledge</b> (2 <sup>nd</sup> and 3 <sup>rd</sup> interviews)
<b>Abel</b>	D-MK1	D-MK3
<b>Mateus</b>	D-MK3	D-MK3/4
<b>David</b>	D-MK1	D-MK2
<b>Frederico</b>	D-MK1	D-MK2

As already stated, this classification is not rigorous, because I did not collect systematic data about their knowledge of the definition during the interviews, in particular, about their final knowledge. Their prior knowledge came out during the 3<sup>rd</sup> seminar and during the 2<sup>nd</sup> and 3<sup>rd</sup> interviews, which showed that, at the beginning of the research process, most of the teachers' knowledge of the  $\varepsilon$ - $\delta$  definition was very limited. All four teachers had been in contact with this definition at some moment during their studies and they faced many difficulties with it. Some of them said that they had just memorised it (Abel, Mateus, and David). The only one who showed himself able to write this definition correctly during the 3<sup>rd</sup> seminar is Mateus.

During this seminar, the discussion about the  $\varepsilon$ - $\delta$  definition lasted about one and a half hours. Several versions of this definition were presented and discussed, with some graphical support. I also tried to make the teachers work with the numerical register, but they were not able to reach any conclusion. I then gave an explanation using several registers, and we discussed the roles of  $\varepsilon$  and  $\delta$  in the definition one more time.

As a consequence of this discussion, the teachers became aware of some specific difficulties with the definition (see next section). However, none of them showed

that he fully understood it. It seems that the discussion within the research group was not sufficient for them to overcome these difficulties.

The  $\varepsilon$ - $\delta$  definition is difficult conceptually. As said before, this definition does not give a direct access to the limit concept. On the contrary, to understand this definition, it is necessary to have a good intuitive understanding of the limit concept, as well as strong knowledge of the use of logical symbols (the quantifiers  $\exists$  and  $\forall$ , and the implication) and of basic mathematical topics such as absolute value and solution of inequalities. For these reasons understanding the  $\varepsilon$ - $\delta$  definition is challenging for most students and teachers.

To fully understand the  $\varepsilon$ - $\delta$  definition, the teachers would probably need more systematic learning, leading them to a deep reflection on it. This is what I began to do during the 3<sup>rd</sup> seminar when I realized that the discussion was not leading to any conclusion: explaining the definition using several registers. This explanation should be followed by solution of tasks in several registers, and more discussions, for them to really gain meaning of the  $\varepsilon$ - $\delta$  definition.

### **11.6.2 Difficulties inherent to the $\varepsilon$ - $\delta$ definition**

Two main difficulties for understanding the definition emerged from the discussions. The first one, pointed out by Ernesto<sup>20</sup> during the third seminar (see Appendix 11.1), relates to the inversion of the steps: the chosen radius  $\varepsilon$  (arbitrarily small) is related to the dependent variable  $y$ , while the radius  $\delta$ , which depends on  $\varepsilon$ , relates to the independent variable  $x$ . This difficulty has been shown by Courant and Robbins (1978), quoted by Fischbein (1993), in the case of limits of sequences (see pages 99-100). The same complexity applies to limits of functions and the  $\varepsilon$ - $\delta$  definition.

The second difficulty was pointed out by Abel during the second interview, when he asked how to use the definition when the function does not have a limit. In fact, the definition does not help to determine a limit, but can be used to prove that a certain value is actually the limit of the function. It seems troublesome for individuals whose main tasks about limits, as students and even as teachers, is

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<sup>20</sup> Ernesto passed away in November 2004 before he concluded his dissertation

calculating limits. They do not understand the need for proof, because they already found the limit through calculations.

### 11.6.3 Teaching the $\varepsilon$ - $\delta$ definition in secondary schools

The classification of teachers' ideas about teaching the  $\varepsilon$ - $\delta$  definition in secondary schools is presented in Table 11.4.

**Table 11-4 Evolution of teachers' ideas about teaching the  $\varepsilon$ - $\delta$  definition in secondary schools**

	<b>Initial ideas</b> (1 <sup>st</sup> interview, 3 <sup>rd</sup> seminar)	<b>Final ideas</b> (2 <sup>nd</sup> and 3 <sup>rd</sup> interviews)
<b>Abel</b>	D-T1	D-T3
<b>Mateus</b>	D-T3	D-T2
<b>David</b>	D-T4	D-T2
<b>Frederico</b>	D-T4	D-T2

A study of this table shows that there is a great discrepancy between the evolution of teachers' ideas about teaching the  $\varepsilon$ - $\delta$  definition in schools. For the other aspects of MfT analysed in previous chapters, teachers' ideas evolved from being close to the secondary school institutional relation to challenging this institutional relation. This is not the case for the teaching of the  $\varepsilon$ - $\delta$  definition, for any of the teachers other than Abel.

During the research process, all teachers showed that they were aware that secondary school students were unable to understand the  $\varepsilon$ - $\delta$  definition. Two of them (David and Frederico) even said during the 1<sup>st</sup> interview that they would not teach it. However, during the third interview, three of them declared that they would teach this definition at secondary school, even if they would not ask students to memorize or apply this definition. For three of them, this means going back to the institutional relation. I can see two interpretations for this fact.

The first one is the difficulty that teachers have in breaking too many of the rules of the institutional relation, of being a “bad subject” (using Chevallard's terms) of the institution. The numerical and graphical registers are mentioned in the syllabus, even if they are not used in practice. Using these two registers in teaching cannot be considered as withdrawing too much from the syllabus. But

the  $\varepsilon$ - $\delta$  definition is also part of the syllabus, even if implicitly (see Chapter 2). Not to teach it could be considered as acting “against” the syllabus, which represents the authority.

The second interpretation is a mathematical one: a concept must have a definition. As explained in Chapter 2 (see pages 17- 18), there are two ecological constraints to the existence of a MO inside an institution: on the one hand it must be understandable; on the other hand it must be justified, in reference to the scholarly mathematical knowledge. In the case of the  $\varepsilon$ - $\delta$  definition, these two constraints are in conflict, hence a dilemma for the teacher.

David expressed clearly these two main reasons for teaching the definition in schools in the following extract from his 3<sup>rd</sup> interview (in this quote, the parts of David’s utterances that I used to draw my conclusions are indicated in bold):

I: Então, havia de começar pelo quadro numérico, não é. E depois? ...

D: É que assim não havia de fugir à regra, não é, porque prontos, esses professores dão a definição

I: Hum, hum, hum. Ah, havia de dar?

D: A definição eu havia de dar!

I: Hum

D: Porque é um conceito, e todo o conceito tem que ser definido... mas, eu particularmente acho que haveria de usar muito pouco (I3/D/2067-79).

I: So, you said that you would start by the numerical setting. And then? ...

D: That way **I wouldn’t get out of the rule**, you know, because you know, these teachers, they teach the definition

I: Hum, hum, hum. Ah, you would teach it?

D: The definition, **I would teach it!**

I: Hum

D: **Because it is a concept, and any concept must be defined** ... but, myself in particular, I think that I would use it very little.

The change of teachers’ personal relation to the  $\varepsilon$ - $\delta$  definition, from challenging the institutional relation to adhering to it, might be explained by a change of their position within the new institution. The first interview was one of their first contacts with me as their supervisor. They did have experience of the research process, and they probably felt like student in front of one of their lecturers. Their opinion about teaching the  $\varepsilon$ - $\delta$  definition in secondary schools reflected a student’s point of view. At the end of the research process, they probably identified themselves more as teachers, on the one hand because of the kind of relationship established within the group, and on the other hand because they were concluding their degree. Even if they were teaching before having concluded their



training, they would now be graduate teachers. Their opinion about teaching the  $\varepsilon$ - $\delta$  definition in secondary schools reflected a teacher's point of view.

This does not apply to Abel, who had already a long teaching experience at the beginning of the process, and had even taught limits in Grade 12. He positioned himself as a more experienced teacher in front of his colleagues, as shown by the way he went to the blackboard to explain the definition during the 3<sup>rd</sup> seminar, as if he were his colleagues' teacher. Abel taught limits according to the institutional relation and, at the end of the research process, is challenging this relation.

The evolution of teachers' personal relation to the  $\varepsilon$ - $\delta$  definition of limits, according to the two components (mathematical knowledge and ideas about teaching) is summarised in Table 11.5.

**Table 11-5 Evolution of teachers' personal relation to the  $\varepsilon$ - $\delta$  definition**

	Mathematical knowledge	Ideas about teaching
<b>Abel</b>	D-MK1 $\rightarrow$ D-MK3	D-T1 $\rightarrow$ D-T3
<b>Mateus</b>	D-MK3 $\rightarrow$ D-MK3 or 4	D-T3 $\rightarrow$ D-T2
<b>David</b>	D-MK1 $\rightarrow$ D-MK2	D-T4 $\rightarrow$ D-T2
<b>Frederico</b>	D-MK1 $\rightarrow$ D-MK2	D-T4 $\rightarrow$ D-T2

These results are discussed in the next chapter which presents the conclusions and limitations of this study.

# CHAPTER 12

## CONCLUSIONS

## 12 Conclusions

This study, based on Chevallard's anthropological theory of didactics, investigates the evolution of four teachers' personal relation to a mathematical concept through their participation in a research group. It shows that their personal relation evolved, but that this evolution had notable limitations.

During this study, I expected to play two roles simultaneously, that of researcher and supervisor of teachers' personal research. I ended up playing three, supervisor, researcher and teacher, due to the teachers' lack of mathematical knowledge.

This led me to a critical analysis of the teachers-as-researchers movement, showing some of the limitations of learning through research.

On a theoretical point of view, this study builds a bridge between two theoretical frameworks: Even's framework for Subject Matter Knowledge (Even, 1990 and 1993), and Chevallard's theories of didactical transposition (1985 and 1991) of anthropological theory of didactics (1999). It presents a new framework for *Mathematics for Teaching* (MfT), applied to the concept *limits of functions*.

On a methodological point of view, choices were made for this study, and the limitations of these choices are reflected upon in this chapter.

Finally, the results of this study also lead me to some suggestions for teacher training, in Mozambique and in other countries.

### **The evolution of teachers' personal relation to limits through the research group**

The evolution of teachers' personal relation to limits according to the five aspects of MfT limits considered in this study is summarised in Table 12.1 (see next page). A critical analysis of this table shows that the evolution of the teachers' knowledge was uneven in most of these aspects.

**Table 12-1 Evolution of teachers' personal relation to the limit concept according to five aspects of *mathematics for teaching***

	Categories	Abel	Mateus	David	Frederico
<b>First Encounter</b>	FE-MK1 to FE-MK2 FE-T1 TO FE-T6	FE-MK1 $\rightarrow$ FE-MK2 FE-T2 $\rightarrow$ FE-T4	FE-MK1 $\rightarrow$ FE-MK2 FE-T3 $\rightarrow$ FE-T6	FE-MK1 $\rightarrow$ FE-MK2 FE-T1 $\rightarrow$ FE-T5	FE-MK1 $\rightarrow$ FE-MK2 FE-T3 $\rightarrow$ FE-T5
<b>Social Justification</b>	SJ-MK1 to SJ-MK4 SJ-T1 TO SJ-T3	SJ-MK2 $\rightarrow$ SJ-MK4 SJ-T1 $\rightarrow$ SJ-T1	SJ-MK1 $\rightarrow$ SJ-MK3 SJ-T1 $\rightarrow$ SJ-T2	SJ-MK1 $\rightarrow$ SJ-MK4 SJ-T1 $\rightarrow$ SJ-T3	SJ-MK2 $\rightarrow$ SJ-MK3 SJ-T1 $\rightarrow$ SJ-T1
<b>Essential Features</b>	EF-MK1 to EF-MK4 EF-T1 TO EF-T2	EF-MK1 $\rightarrow$ EF-MK4 EF-T1 $\rightarrow$ EF-T2	EF-MK3 $\rightarrow$ EF-MK4 EF-T1 $\rightarrow$ EF-T2	EF-MK2 $\rightarrow$ EF-MK4 EF-T1 $\rightarrow$ EF-T2	N/A EF-T1 $\rightarrow$ EF-T2
<b>Graphical Register</b>	GRRR to GRR6 GRRS to GRS6 GR-T1 to GR-T3	GRR1 $\rightarrow$ GRR3 GRS2 $\rightarrow$ GRS2 GR-T1 $\rightarrow$ GR-T2	GRR5 $\rightarrow$ GRR6 GRS3 $\rightarrow$ GRS3 GR-T1 $\rightarrow$ GR-T3	GRR2 $\rightarrow$ GRR5 GRS3 $\rightarrow$ GRS6 GR-T1 $\rightarrow$ GR-T3	GRR3 $\rightarrow$ GRR4 GRS1 $\rightarrow$ GRS4 GR-T1 $\rightarrow$ GR-T3
<b><math>\varepsilon</math>-<math>\delta</math> Definition</b>	D-MK1 to D-MK4 D-T1 to D-T4	D-MK1 $\rightarrow$ D-MK3 D-T1 $\rightarrow$ D-T3	D-MK3 $\rightarrow$ D-MK3/4 D-T3 $\rightarrow$ D-T2	D-MK1 $\rightarrow$ D-MK2 D-T4 $\rightarrow$ D-T2	D-MK1 $\rightarrow$ D-MK2 D-T4 $\rightarrow$ D-T2

The *essential features of the limit concept*, a sub-category of the *scholarly mathematical knowledge*, is the only aspect in which the evolution of teachers' knowledge evolved in a similar way. Two reasons can explain this fact. The first is that this aspect was not directly linked to any specific research topic. This means that the four teachers were in a similar position regarding this aspect, learning about it through their reading of research papers in mathematics education and the discussions within the research group, and not directly through their own research. The second reason is that this aspect only involves general mathematical knowledge. Understanding that the limit concept can be seen from a dynamic, a static or an operational point of view does not require a deep knowledge of mathematical concepts. From the point of view of their mathematical knowledge, the teachers were also in a similar situation regarding this aspect. For these reasons, at the end of the research process, all four teachers knew that the limit concept can be seen from different points of view and were willing to show these different features to their students.

Two other aspects of MfT limits, how to organise students' *first encounter* with the limit concept (Chapter 7) and the *social justification* for teaching limits in secondary schools (Chapter 8), were directly linked with two research topics (Abel's and David's topics, respectively), and also involved general mathematical knowledge. For both aspects, the evolution of teachers' personal relation appears to be uneven, depending not only on the teachers' research topic, but also on how they positioned themselves within the group, more as teacher-learners or as learner-teachers.

The influence of these different positions is even more visible in the use of the *graphical register* for studying limits (Chapter 10). Furthermore, in this aspect of MfT limits, the evolution of teachers' knowledge appears to be more limited than in the three previous aspects. Teachers' difficulties in working with graphs have been explained in relation to a lack of deep understanding of certain basic mathematical concepts. While the learning of the three previous aspects only requires general mathematical knowledge, working with graphs involves using more specific mathematical knowledge, which all teachers had difficulties with, although at different levels.

The analysis of the evolution of their personal relation to the  $\varepsilon$ - $\delta$  *definition* (Chapter 11), which not only requires deep understanding of basic mathematical concepts but is also intrinsically difficult, confirms this analysis. Also, for the three teachers who had never taught limits in Grade 12, while their personal relation to limits in the previous four aspects evolved from being close to the secondary schools' institutional relation to being closer to the relation of the new institution, their ideas about teaching the  $\varepsilon$ - $\delta$  definition in secondary schools evolved in the opposite direction. This has been explained by the evolution of their position within the new institution, from learner-teachers to teacher-learners. The end of their dissertation was also the conclusion of their teacher training and they positioned themselves more as teachers than as students, looking forward to their new institution, the secondary school. When teaching limits in schools, they will have to face the conflict between two ecological constraints: the knowledge to be taught must be understandable, but must also be justified in reference to the scholarly mathematical knowledge. They know that secondary school students usually do not understand the  $\varepsilon$ - $\delta$  definition, but it is part of the Grade 12 syllabus, and they remember studying it in that grade. They were probably feeling more strongly the weight of the secondary schools' institutional relation to limits at the end of the research process.

In conclusion, the analysis of the evolution of these teachers' personal relation to limits, with respect to five categories or sub-categories of *mathematics for teaching* limits, shows that their knowledge evolved in significant ways for most of these aspects, from being close to the Mozambican didactic institutions' relation to limits to being closer to the new institution's relation to this concept. However, although the new institution had a strong impact on teachers' personal relation to limits, two limitations became evident during this study.

The first limitation refers to the kind of teachers receptive to change through a research group. Within the research group set up for this study, two extreme positions appeared.

On the one hand Abel, who had taught limits in schools for many years, positioned himself as an experienced teacher from the beginning of this study. For example, he did not try to solve any of the tasks presented during the 1<sup>st</sup> interview, and tried to focus on

whether these tasks were used in schools or not, and at the beginning of the 3<sup>rd</sup> seminar, he went to the blackboard to explain the  $\varepsilon$ - $\delta$  definition to his colleagues, as if they were his students. The challenge to the content of his teaching that occurred during the research, either as an experienced Grade 12 teacher or as a researcher during his experiment, put him in an uncomfortable position, which may have hindered the evolution of his personal relation to limits.

On the other hand David, the youngest of the group and with very limited teaching experience, assumed more of a student position. When speaking about the first encounter during the first interview, he adopted the students' point of view while his colleagues adopted a teacher's position. During the 12<sup>th</sup> seminar, he stood up for the numerical method, arguing that students face difficulties in working with graphs, while Mateus and Frederico defended that, as teachers, they should teach graphical methods. During the 3<sup>rd</sup> interview, he spontaneously wrote down the  $\varepsilon$ - $\delta$  definition of limits, and tried to solve many graphical tasks, using my explanations to solve more tasks. As a result, David was more receptive to learning the mathematical content and challenging the secondary schools' institutional relation to limits. This is particularly notable for the graphical register.

The second limitation is regarding the choice of the mathematical topic to be researched by the teachers. When deep understanding of related mathematical concepts is lacking, or when the mathematical content is conceptually difficult as in the case of the  $\varepsilon$ - $\delta$  definition of limits, learning through a research institution might not be sufficient for the teachers to overcome their mathematical difficulties. I suggest that, in such a case, a more direct teaching, involving explanations, solution of tasks using different registers, and discussions, might be more productive.

This suggests that teachers should research the institutional relation of topics of which they have a strong mathematical knowledge, or that the research institution should have a teaching component, as suggested by the analysis of the researcher-supervisor role presented below.

### **My double role: researcher and supervisor**

Another issue that I had intended to analyse throughout my research was my double role as the teachers' supervisor in their personal research, and simultaneously as a researcher of the evolution of their personal relation to limits of functions, as explained in the methodological chapter (see pages 167-168). Although I initially started my journal for this purpose, very few entries refer to this supervisor-researcher dichotomy.

During the first interview, I observed that these two roles were well defined in terms of time: for each of the teachers, I first held the interview, and then stopped recording to go on with the supervision session. This occurred naturally for all teachers except for Ernesto, who tried several times to speak about his research during the interview, and to whom I had to explain that this would be discussed afterwards.

As a researcher, I was worried that the teachers would drop out of the group, which would hinder the progress of my study. This led me to contact the teachers when they did not appear for a long time. For example, on the 18<sup>th</sup> of January 2003, I wrote:

I haven't heard from the UP students since November. I hesitated to send them a message. As their supervisor, I think that they should contact me. However I feel that my research group did not really come to life. The first seminar was successful, but not enough to create a team spirit. I think that we need to have more frequent seminars, maybe monthly (My journal, page 24)

In that entry, my main concern was my research, even if there was no real conflict with my role as supervisor. As a consequence, I contacted all the teachers, which I probably would not have done had I just been a supervisor.

On the 21<sup>st</sup> of June 2003, I wrote a comment about my double role, pointing out that there was no contradiction between these two roles. It reads:

I realise that I began writing this journal to analyse my double role: supervisor and researcher. However, I have written very little about that because I do not feel any contradiction between these two roles. I believe things are going rather smoothly. I also think that any supervisor is also observer of her students' evolution of knowledge. In this case the difference is that this observation is more systematic (My journal, page 70)



In February 2003 I faced a dilemma. Two students had already finished their dissertations, while two others were far from reaching that point. The research group could hardly continue with only two teachers, and my data collection was running late. Fortunately, Mateus and Frederico, who had already defended their dissertation, accepted to attend two more seminars. I interviewed them during the period between these seminars. However, I did not wait for David and Abel to conclude their dissertation to interview them for the third time. In this case there was a small time-conflict between the supervision and my own research.

In April 2003, after the 13<sup>th</sup> and last seminar, where we discussed the graphical register, I observed that during the seminar I sometimes acted more as a teacher than as a moderator of the discussion. I wrote: “as I am reluctant to do that, this made my task difficult. For example, I did not check that all of them were able to solve the tasks” (My journal, page 98)

I then explained my dilemma.

When I began writing this journal, my main objective was to analyse my double role as supervisor-researcher and possible problems which arose from that. I now realise that in fact my role is triple (supervisor, researcher, and teacher) because of the teachers' lack of mathematical knowledge. Having to act as a teacher causes me more problems because I feel that this distorts our relationship (My journal, page 98)

I remember that this also happened during the 3<sup>rd</sup> seminar, when we discussed the  $\varepsilon$ - $\delta$  definition, although I have no record of this in my journal. As a researcher, during that seminar I acted more as a moderator and observer, trying not to influence the teachers' own ideas. For this reason, I did not explain the definition, even though I could see that they were not reaching any conclusion. I began acting as a teacher after one and a half hours of seminar, explaining the definition and giving examples in different registers. Obviously this teaching was not systematic. As a conclusion, as with the graphical register, the teachers did not fully understand the  $\varepsilon$ - $\delta$  definition.

To sum up, I would say that instead of a double role (supervisor and researcher), I unexpectedly ended up playing a triple role (supervisor, researcher, and teacher), because of the teachers' weak understanding of some mathematical knowledge. The

main conflict appeared to be between my researcher and teacher roles, because I expected the teachers to have a deeper knowledge of basic mathematical concepts. This fact helped me analyse the teachers-as-researchers movement and its possible impact in the Mozambican context.

### **Teachers-as-researchers**

In Chapter 5, I described several experiences belonging to the teachers-as-researchers movement. I did not find any theoretical study on the field of teachers-as-researchers, but only descriptions of some experiences.

The analysis of these experiences showed that, when choosing some topic to research regarding their own teaching, teachers usually choose pedagogical issues or students' difficulties. It seems that the teachers-as-researchers movement assumes that the teachers master the mathematical content, possibly because this movement developed in contexts in which teachers have a strong mathematical knowledge. In other countries such as Mozambique, where the teachers' mathematical knowledge is not so strong, this kind of research needs to be slightly modified, putting mathematics at the centre.

Furthermore, by putting some teachers in a position where they have to research both mathematical and pedagogical issues, this study suggests that for experienced teachers, who are expected to be experts in the mathematics they have to teach, challenging their own mathematical knowledge is very difficult. This does not seem to be the case for in-training teachers, who are still learning mathematics.

This hypothesis is mainly based on the case studies of two teachers, holding extreme positions within the research group. Hence, it cannot be extended to other teachers, but should be the object of further research, in Mozambique and in other countries.

If confirmed by further research, this hypothesis could also explain why teachers researching their own practice seem to prefer to look at pedagogical issues or students' difficulties than at the mathematical content. In that way they do not need to challenge their own personal relation to mathematics.

This also suggests that learning through research could have a limited impact on experienced teachers, unless they already have strong mathematical knowledge.

## Theoretical contribution

In their introduction to a paper entitled *Twenty-Five Years of Didactic Transposition* (Bosch & Gascón, 2006), Artigue and Hodgson argue:

Building bridges between theoretical approaches and frames, looking for possible connections and complementarities, identifying common concerns but also potential incompatibilities, become more and more a necessity of the research agenda (Artigue & Hodgson, 2006: 50)

I fully agree with this statement. The French-speaking community of researchers in mathematics education, particularly Guy Brousseau and Yves Chevallard, have developed strong theoretical frameworks, which allow a deep analysis of several aspects of the teaching and learning of mathematics. According to Bosch & Gascón (2006), this new paradigm spread quickly in the Spanish-speaking community, but very slowly in the English-speaking community. This can be explained by the fact that very little dissemination of these frameworks has been done in English, which hindered the researchers of the English-speaking community who showed interest in knowing this paradigm in using it.

This study builds a bridge between two theoretical frameworks: Even's framework for Subject Matter Knowledge (SMK) (Even, 1990 and 1993), based on Schulman's distinction of SMK and PCK (Pedagogical Content Knowledge) (Shulman, 1986 and 1987) and Chevallard's theories of didactical transposition (Chevallard, 1985) and of anthropology of didactics (Chevallard, 1999).

The use of Even's framework was important as a starting point of this study to analyse the knowledge that a teacher needs in order to teach limits in secondary schools. However, this analysis showed a gap in this framework: it is mainly descriptive, presenting a list of knowledge components that a teacher needs in order to teach a specific concept, but this list is not well theorised and the distinction between PCK and SMK is sometimes blurred.

A deep reflection on Even's work, through the lens of a strong theoretical framework, Chevallard's theories of didactical transposition and of the anthropology of didactics,

led to the elaboration of a new framework for *mathematics for teaching* a mathematical topic. This new framework does not distinguish mathematical knowledge from pedagogical knowledge. It emerges from the analysis of the knowledge that a teacher would need to be able to consciously take the second step of the didactical transposition (Mathematical knowledge to be taught  $\rightarrow$  Mathematical knowledge actually taught), looking forward to the classroom situation, looking back on both the scholarly mathematical knowledge and on the social justification to teach a specific topic in a specific institution and at a specific level, not merely reproducing the institutional relation.

This framework was used to design the research project and analyse the evolution of teachers' personal relation to limits of functions through their participation in a new institution: the research group.

Furthermore, the process of data analysis led to a new distinction of mathematical knowledge and pedagogical ideas about teaching. Instead of classifying each aspect of MfT limits as *mathematical* or *pedagogical*, each aspect of MfT is rather considered as having two components: mathematical knowledge, and ideas about teaching. Some of the aspects of MfT are more mathematical, others are more pedagogical, but all aspects have these two components. Obviously teachers' ideas about teaching according to each of these aspects deeply depend on their mathematical knowledge but they are also influenced by other issues, in particular their personal experiences as students and as teachers. Their ability to challenge the institutional relation also deeply depends on their previous experiences.

### **Limitations of this study**

The main limitations of this study relate to methodological issues, in particular the set up of the new institution and the process of data collection.

The analysis of the evolution of teachers' personal relation to limits with respect to five categories of mathematics for teaching limits in schools brings to light some limitations due to the teachers' mathematical knowledge of limits and of related mathematical concepts.

At the beginning of the research, I expected the teachers to have a better knowledge about limits of functions. I realised during the first interview that they had difficulties with the  $\varepsilon$ - $\delta$  definition and consequently held a seminar on this issue. I was convinced that a discussion about the definition would be sufficient to overcome their difficulties and only realised some time later that this was not the case. I then gave them an explanation not sufficiently detailed for them to overcome their difficulties.

Regarding the graphical register, I realised late on the study that the teachers were not able to use graphs, and held a seminar about this issue at the end of the process, also giving them the notes that I use with my first-year biology students to explain how to link limits with graphs. I then assumed that this should be enough for them to overcome their difficulties, which was not the case. My reluctance in playing the teacher's role prevented me from teaching the mathematical content when it was necessary.

Reflecting on this issue led me to the conclusion that the new institution needed to take the mathematical start of the teachers more carefully into account, and include some teaching when necessary. This implies that the analysis of the teachers' previous knowledge should be more systematic, which in turn led to reflect on the process of data collection.

For some aspects of MfT limits, data collection was done during the interviews through tasks presented to the teachers, and the observation of their approach to solving these tasks. This is the case in the use of graphs in the study of limits. For other aspects, data was collected in conversation with the teachers during the interviews, or between teachers during the seminars. These differences in data collected for different aspects of MfT caused two limitations. The first consequence, as explained above, is that I did not realise that some teachers' difficulties were so deep, and consequently did not organise systematic activities to overcome these difficulties. The second consequence is that, for some aspects of MfT limits I did not have enough data to analyse. This is the case of the algebraic and numerical registers, which were left out of my study because I considered that they were not as problematic as other aspects. This does not mean that these aspects are not important. As shown throughout this study, the numerical register is hardly used in Mozambican schools, and the algebraic register is often used in a very algorithmic

way. Teachers need more flexibility in the use of these two registers, and I suggest that more attention should be given to them in further studies.

As explained in the methodological chapter, the choice of using interviews instead of questionnaires was based on the fact that I wanted to create a good relationship with the teachers at the beginning of the research process (see page 164). I still believe that it was a good option, because the first interview actually helped me create this initial relationship with the teachers: we knew each other better, and they had a better understanding of the research process. However this interview could have collected data in a more systematic manner, for example by asking to the teacher to write down and explain the  $\varepsilon$ - $\delta$  definition.

At the time of the third interview, held at the end of the research process, I had already worked with the teachers for more than a year, we already knew each other and had established a good relationship. Furthermore we spent a lot of time during this last interview solving graphical tasks, which limited the discussion on other issues. I now believe that this interview should have been preceded by a questionnaire, where data collection about all aspects of MfT limits would have been more systematic. I could have designed tasks or specific questions for each aspect of MfT limits. This questionnaire would not take the place of the interview, because many aspects of teachers' points of view would not be formulated through a questionnaire. However the interview could be based on teachers' answers to the questionnaire, which would enable a deeper analysis of teachers' personal relation to limits at the end of the research process.

To sum up, in this kind of research, the research institution should be designed taking the previous mathematical knowledge of the teachers involved more carefully into account, and organising systematic activities for them to overcome their difficulties.

### **Suggestions for teacher training**

This study points out some limitations of teacher training. In Mozambique, as in many other countries, the mathematical and the pedagogical knowledge is learnt in different courses, as in the *knowledge-for-practice* conception. Students are required to take

several mathematics courses before taking the pedagogical ones. The mathematics courses are taught in the same way as for pure mathematics or applied mathematics students. Reference to teaching is hardly done. Students must then take some pedagogical courses, and are expected to make the link between their mathematical and pedagogical knowledge.

I suggest that, in teacher training courses, mathematics should be taught in a different way, including the aspects of mathematics for teaching presented in this study. In that way teachers could begin to reflect on the institutional relation to mathematics of secondary schools from the beginning of their training.

Furthermore, I suggest that student-teachers be involved in research putting mathematics at the centre during their training: research on *mathematics for teaching*, according to the framework developed in this study, and based on both mathematical and pedagogical issues. In this way they will produce knowledge that helps them evolve their personal relation to mathematics and its teaching and learning, as well as hopefully improve their practice. Obviously I do not claim that they would necessarily teach differently, as they would be exposed to institutional constraints, but that their personal relation to mathematics would at least enable them to teach in a more elaborated way.

For example, for their dissertation, student-teachers could be involved in research on mathematical concepts, studied from the teacher's point of view, using the aspects of *mathematics for teaching* elaborated in this study. In that way, student-teachers would not be considered as mere consumers of mathematical knowledge, as in the *knowledge-for-practice* conception, but as producers of mathematical knowledge, in accordance with the *knowledge-of-practice* conception adopted in this study.

*Mathematics for teaching* should be at the centre of teachers' research, both for teachers in service and for teachers in training.

# REFERENCES



## REFERENCES

- Adler, J. & Davis, Z. (2006) Opening another Black Box: Researching Mathematics for Teaching in Mathematics Teacher Education, *Journal for Research in Mathematics Education*, vol.36, no.4, pp.270-296.
- Adler, J., Davis, Z., Kazima, M., Parker, D. & Webb, L. (2005) Working with Learners' Mathematics: Exploring a Key Element of Mathematical Knowledge for Teaching, In: Chick, H. L. & Vincent, J. L. eds. *Proceedings of the 29<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education*, vol.2, pp.1-8.
- Adler, J. (2002) Global and local challenges of teacher development, In: Adler, J. & Reed, Y. eds. *Challenges of teacher development – An investigation of take-up in South Africa*, Pretoria: Van Schaik Publishers, pp.1-16.
- Adler, J. (1992) Action Research and the Theory-Practice Dialectic: Insights from a Small Post Graduate Project Inspired by Activity Theory, In: Geeslin, W. & Graham, K. eds. *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education*, vol.1, pp. 41-48.
- Arsac, G. (1992a) The evolution of a theory in didactics: the example of didactic transposition, In: Douady R. & Mercier A. eds. *Recherches en Didactique des Mathématiques, Selected Papers*, Grenoble: La Pensée Sauvage, pp.107-130.
- Arsac, G. (1992b) Reproducibility of didactical situations, *Educational Studies in Mathematics*, vol.23, pp.5-29.
- Arsac, G. (1989) Le rôle du professeur -. Aspects pratiques et théoriques, reproductibilité. *Cahiers du Séminaire de Didactique des mathématiques et de l'informatique*, Grenoble : IMAG-LSD.
- Artigue, M. & Hodgson, R. (2006) Building Birdges between Theoretical Frameworks in Mathematics Education, *ICMI Bulletin No. 58*, pp. 50-51
- Artigue, M. (1992) Didactical Engineering, In: Douady R. & Mercier A. eds. *Recherches en Didactique des Mathématiques. Selected Papers*, Grenoble: La Pensée Sauvage, pp.41-65.
- Ávila, G. (1998) *Introdução ao Cálculo*, Rio de Janeiro: LTC, Ed. S.A.
- Ball, D., Bass, H. & Hill, H. (2004) Knowing and using mathematical knowledge in teaching: learning what matters, In: Buffler, A. & Laugksch, R. eds. *Proceedings of the 12<sup>th</sup> Annual Conference of the SAARMSTE*, pp.51-65.
- Barbé, J., Bosch, M., Espinoza, L. & Gascón, J. (2005) Didactic restrictions on the teacher's practice: the case of limits of functions in Spanish high schools, *Educational Studies in Mathematics*, vol. 59, pp.235-268.
- Berquembauvev, E., Cherbakov, E., Mozolevski, I., Evdoquimov, M., Gerdes, P. & Alexandrov, R. (1981) *Matemática, Manual da 11a classe, Volume I, Introdução à*

- Análise Matemática: limites e derivadas*, Maputo: Universidade Eduardo Mondlane, Faculdade de Matemática.
- Bkouche, R. (1996) Point de vue sur l'enseignement de l'analyse: des limites et de la continuité dans l'enseignement, *Repères-Irem*, vol.24. Pont-à-Mousson: Topiques Editions, pp.67-76.
- Boero, P., Daputo, C. & Parenti, L. (1996) Didactics of Mathematics and the Professional Knowledge of Teachers, In: Bishop, A., Clements, K., Keitel, C., Kilpatrick, J. & Laborde, C. eds. *International Handbook of Mathematics Education*. Dordrecht: Kluwer Academic Publishers, pp.1097-1120
- Bosch, M. & Gascón, J. (2006) Twenty-Five Years of the Didactic Transposition, *ICMI Bulletin No. 58*, pp. 51-65
- Bosch, M. & Gascón, J. (2002) Organiser l'étude. 2. Théories et empiries, In : Dorier J.-L., Artaud M., Artigue M., Berthelot R., & Floris R. eds. *Actes de la 11<sup>e</sup> École d'Été de Didactique des Mathématiques*, Version électronique du cédérom d'accompagnement, Grenoble : La Pensée Sauvage.
- Bosch, M. & Chevallard, Y. (1999) Sensibilité de l'activité mathématique aux ostensifs, *Recherches en Didactique des Mathématiques*, vol.19, no.1, pp.77-123.
- Breen, C. (2003) Mathematics Teachers as Researchers: Living on the Edge, In: Bishop, A., Clements, M., Keitel, C., Kilpatrick, J. & Leung, F. eds. *Second International handbook of Mathematics Education*, Dordrecht: Kluwer Academic Publishers, pp.523-544.
- Breen, C. (1997) Teachers as Researchers? In: Zack, V., Mousley, J. & Breen, C. eds. *Developing Practice Teachers' Inquiry and Educational Change*, Geelong Australia: Centre for Studies in Mathematics, Science and Environmental Education, Deakin University, pp.151-158.
- Brousseau, G. (1998) *Théorie des situations didactiques*, Grenoble: La Pensée Sauvage.
- Brown, D. (1997) Teachers as Researchers: Some Historical Issues, In: Zack, V., Mousley, J. & Breen, C. eds. *Developing Practice Teachers' Inquiry and Educational Change*, Geelong Australia: Centre for Studies in Mathematics, Science and Environmental Education, Deakin University, pp.193-202.
- Chevallard, Y. (2002a) Organiser l'étude. 1. Structures & fonctions, In : Dorier, J.-L., Artaud, M., Berthelot, R. & Floris, R. eds *Actes de la 11<sup>e</sup> École d'Été de Didactique des Mathématiques*, Version électronique du cédérom d'accompagnement, Grenoble : La Pensée Sauvage.
- Chevallard, Y. (2002b) Organiser l'étude. 1. Ecologie & régulation, In : Dorier, J.-L., Artaud, M., Berthelot, R. & Floris, R. eds. *Actes de la 11<sup>e</sup> École d'Été de Didactique des Mathématiques*, Version électronique du cédérom d'accompagnement, Grenoble : La Pensée Sauvage.
- Chevallard, Y. (1999) L'analyse des pratiques enseignantes en théories anthropologiques du didactique, *Recherches en Didactique des Mathématiques*, vol.19, no.2, pp.221-265.

- 
- Chevallard, Y. (1995) La fonction professorale: esquisse d'un modèle didactique, In : Noirfalise, R. & Perrin-Glorian, M.J. eds. *Actes de l'Ecole d'Été de Didactique des Mathématiques*, Saint-Sauves d'Auvergne, pp.83-122.
- Chevallard, Y. (1992) Fundamental concepts in didactics: perspectives provided by an anthropological approach, In: Douady R. & Mercier A. eds. *Recherches en Didactique des Mathématiques : Selected Papers*, Grenoble: La Pensée Sauvage, pp.131-167.
- Chevallard, Y. (1991) *La transposition didactique - du savoir savant au savoir enseigné*, 2<sup>ème</sup> édition avec un exemple de transposition didactique de Yves Chevallard et Marie-Alberte Joshua, La Pensée Sauvage.
- Chevallard, Y. (1985) *La transposition didactique - du savoir savant au savoir enseigné*, Grenoble: La Pensée Sauvage.
- Choquet, G. (1966) *Topology*. New York: Academic Press.
- Cochran, K., DeRuiter, J. & King, R. (1993) Pedagogical Content Knowing: An Integrative Model for Teacher Preparation, *Journal of Teacher Education*, vol.44, no. 4: pp.263-271.
- Cochran-Smith, M. & Lytle, S. (1999) The Teacher Research Movement: A Decade Later, *Educational Researcher*, vol.28, no.7, pp.15-25.
- Cochran-Smith, M. & Lytle, S. (1998) Relationships of Knowledge and Practice: Teacher Learning in Communities, *Review of Research in Education*, vol.24, pp.249-305.
- Cochran-Smith, M. & Lytle, S. (1990) Research on Teaching and Teacher Research: The Issues That Divide, *Educational Researcher*, vol.19, no.2, pp.2-11.
- Cohen, L. & Manion, L. (1985) *Research Methods in Education*, Second edition, London: Croom Helm.
- Cornu, B. (1991) Limits, In: Tall, D. ed. *Advanced Mathematical Thinking*, Dordrecht: Kluwer Academic Publishers, pp.153-166.
- Cornu, B. (1984) Interférence des modèles spontanés dans l'apprentissage de la notion de limite, *Séminaire de Recherche Pédagogique IMAG*, pp.236-268.
- Cornu, B. (1983) *Apprentissage de la notion de limite: conceptions et obstacles*, Thèse de doctorat de troisième cycle, Grenoble : Université Scientifique et Médicale.
- Cornu, B. (1982) Grandes lignes de l'évolution historique de la notion de limite, *Bulletin APMEP*, vol.335, pp.627-641.
- Costa, M.J. (1998) *Vizualização e Mudança de Quadros numa Perspectiva Construtivista: uma Contribuição para o Estudo das Inequações*, Trabalho de Diploma, Maputo: Universidade Pedagógica.
- Courant, R. & Robbins, H. (1978) *What is mathematics? An elementary approach to ideas and methods*, Oxford : Oxford University Press.

- 
- Crawford, K. & Adler, J. (1996) Teachers as Researchers in Mathematics Education, In: Bishop, A. et al eds. *International Handbook of Mathematics Education*, Dordrecht: Kluwer Academic Publishers, pp.1187-1205.
- D'Ambrosio, B. (1998) Using Research as a Stimulus for Learning, *Journal for Research in Mathematics Education: Qualitative Research Methods in Mathematics Education*, Monograph n° 9, pp.145-177.
- Dahan-Dalmedico, A. & Peiffer, J. (1986) *Une histoire des Mathématiques – Routes et dédales*, Paris: Editions du Seuil.
- Demidovitch, B. (1977) *Problemas e exercícios de Análise Matemática*, Moscou: Editora Mir.
- Douady, R. (1986) Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, vol.7, no.2, pp.5-31.
- Duval, R. (1999) *Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning*,  
<http://peabody.vanderbilt.edu/depts/tandl/mted/thompson/PDFversions/PME99Duval.pdf>
- Duval, R. (1996) Quel cognitif retenir en Didactique des Mathématiques, *Recherches en Didactique des Mathématiques*, vol.16, no.3, pp. 349-380.
- Edwards, C.H. (1937) *The Historical Development of the Calculus*, New York: Springer-Verlag.
- Edwards, T. & Hensien, M. (1999) Changing Instructional Practice through Action Research, *Journal of Mathematics Teacher Education*, vol.2: pp.187-206
- Elliott, J. (1991) *Action Research for Educational Change*, Milton Keynes, Philadelphia: Open University Press.
- Even, R. (1993) Subject-Matter Knowledge and Pedagogical Content: Prospective Secondary Teachers and the Function Concept, *Journal for Research in Mathematics Education*, vol.24, no.2, pp.94-116.
- Even, R. (1990) Subject Matter Knowledge for Teaching and the Case of Functions, *Educational Studies in Mathematics*, vol.21, pp.521-544.
- Farah-Sarkis, F. (1999) Inservice in Lebanon, In: Jaworski, B, Wood, T. & Dawson, S. eds. *Mathematics Teacher Education: Critical International Perspectives*, London: Falmer Press, pp.43-47.
- Fischbein, E. (1993) The Interaction between the Formal, the Algorithmic, and the Intuitive Components in a Mathematical Activity, In: Biehler, R., Scholz, R.W., Sträber, R. & Winkelmann, B. eds. *Didactics of Mathematics as a Scientific Discipline*, Dordrecht: Kluwer, pp.231-245.
- Graven, M. (2005) Dilemmas in the design of in-service education and training for mathematics teachers, In: Vithal, R., Adler, J. & Keitel, C. eds. *Researching Mathematics Education in South Africa – Perspectives, practices and possibilities*, Cape Town: HSRC Press, pp.206-232.

- 
- Graven, M. (2002) *Mathematics Teacher Learning, Communities of Practice and the Centrality of Confidence*, PhD Thesis, Johannesburg: University of the Witwatersrand.
- Halai, A. (1999) Mathematics Education Research Project: Researching Teacher Development through Action Research, In: Zaslavski O. ed. *Proceedings of the 23<sup>rd</sup> PME Conference*, vol.3, pp.65-72.
- Hatch, G. & Shiu, C. (1998) Practitioner Research and the Construction of Knowledge in Mathematics Education, In: Sierpinska, A. & Kilpatrick, J. eds. *Mathematics Education as a Research Domain: A Search for Identity*, Dordrecht: Kluwer Academic Publishers, pp.297-315.
- Hitchcock, G. & Hughes, D. (1989) *Research and the Teacher: a qualitative introduction to school-based research*, 2<sup>nd</sup> edition, New York: Routledge.
- Huillet, D. (2000a) *Análise Matemática – Agronomia*. Maputo: U.E.M. Departamento de Matemática e Informática.
- Huillet, D. (2000b) A modelização intramatemática: exemplo das funções racionais, *Matemática, Estatística, Informática*, no.6, Maputo: U.E.M. Departamento de Matemática e Informática
- Huillet, D. & Mutemba, B. (2000) The Relation of Mozambican Secondary School Teachers to a Mathematical Concept: the Case of Limits of Functions, In: Mahlomaholo, S. ed. *Proceedings – 8<sup>th</sup> Annual Meeting SAARMSE Conference*, pp.309-316.
- Hoffman, L. & Bradley, G. (1996) *Calculus for Business, Economics, and the Social and Life Sciences*, Sixth edition, Boston: Mac Graw Hill.
- Iezzi, G., Murakami, C., Hazzan, S. & Pompeu, N.J. (1985) *Fundamentos de Matemática Element: Limites, Derivadas, Noções de Integral – Volume 8*, São Paulo: Atual Editora.
- Janvier, C. (1987) Translation Processes in Mathematics Education, In: Janvier C. ed. *Problems of Representation in the Teaching and Learning of Mathematics*, Hillsdale, New Jersey: Lawrence Erlbaum Associates, pp.27-32.
- Jaworski, B. (1998) Mathematics Teacher Research: Process, Practice and the Development of Teaching, *Journal of Mathematics Teacher Education*, no.1, pp.3-31.
- Kvale, S. (1996) *Interviews - An Introduction to Qualitative Research Interviewing*, Thousand Oaks: Sage Publication.
- Laborde, C. (1992). Problèmes de compréhension de textes mathématiques au collège, *Séminaire R2I. Recherche/Réflexion/Interaction*, no.1. Grenoble: I.U.F.M. de Grenoble, pp.7-27.
- Larson, R., Hostetler, R. & Edwards, B. (1994) *Calculus of a single variable*, Lexington: D. C. Health and Company.

- Legrand, M. (1996) La problématique des situations fondamentales, *Recherches en Didactique des Mathématiques*, vol.16, no.2, pp.221-279.
- Long, C. (2003) Mathematics knowledge for teaching: how do we recognise this? In: Jaffer, S. & Burgess, L. eds. *Proceedings of the Ninth National Congress of the Association for Mathematics Education in South Africa (AMESA)*.
- Margolinas, C., Coulange, L. & Bessot, A. (2005) What can the teacher learn in the classroom? *Educational Studies in Mathematics*, vol.59, pp.205-234.
- Maxwell, J. (1992) Understanding and Validity in Qualitative Research. *Harvard Educational Review*, vol.62, no 3, pp.279-300.
- Miles, B. & M., Huberman, A (1994) *Qualitative Data Analysis: an expanded sourcebooks*, 2<sup>nd</sup> edition, London: Sage Publication.
- Ministério da Educação (1997) Programas de Matemática - 2º ciclo. 12ª classe, Maputo.
- Ministério da Educação (1993) Programas de Matemática - 2º ciclo. 12ª classe, Maputo.
- Monaghan, J. (1991) Problems with the Language of Limits, *For the Learning of Mathematics*, vol.11, pp.20-24.
- Morettin, P., Hazzan, S. & Bussab, W. (2003) *Cálculo – Funções de uma e várias variáveis*, São Paulo: Saraiva Editora.
- Mousley, J. (1992) Teachers as researchers: Dialectics of Action and Reflection, In: Geeslin, W. and Graham, K. eds. *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education*, vol.2, pp.136-143.
- Mutemba, B. (2001) *The Mozambican Students' Understanding of the Concept Limit of a Function: a Case Study*, Unpublished Masters Degree Dissertation, Johannesburg: Witwatersrand University.
- Mutemba, B. & Huillet, D. (1999) Institutional Relation to a Mathematical Concept: the Case of Limits of Functions in Mozambique, In: Kuiper, J. ed. *Proceedings – 7<sup>th</sup> Annual Meeting SAARMSE Conference*, pp.191-196.
- Piscounov, N. (1977) *Cálculo Diferencial e Integral, Volume 1*, Porto: Edições Lopes da Silva.
- Protter, M. & Morrey, C. (1977) *Calculus with Analytic Geometry. A first course*. Berkeley: Edition Addison-Wesley Publishing Company.
- Richardson, V. (1994) Conducting Research on Practice, *Educational Researcher*, vol.23, no.5, pp.1-10.
- Robert, A & Robinet, J (1989) Représentations des enseignants de mathématiques sur les mathématiques et leur enseignement, *Cahier de DIDIREM*, no 1, Paris : IREM Paris 7.
- Robinet, J. (1983) Une expérience d'ingénierie didactique sur la notion de limite de fonction. *Recherches en Didactique des Mathématiques*, vol.4, no.3, pp.223-292.

- 
- Rossouw, L. & Smith, E. (1998) Teachers' Pedagogical Content Knowledge of Geometry, In: Alwyn, O. & Newstead, K. eds. *Proceedings of the 22<sup>nd</sup> Conference of PME*, vol.4, pp.57-64
- Schneider, M. (2001) Praxéologies didactiques et praxéologies mathématiques à propos d'un enseignement des limites au secondaire. *Recherches en Didactique des Mathématiques*, vol.21, no.1/2, pp.7-56.
- Setati, M (2000) Classroom-based Research: from *with* or *on* teachers to *with* and *on* teachers. Unpublished paper. Johannesburg: Wits University.
- Sierpinska, A. (1987) Humanities Students and Epistemological Obstacle Related to Limits. *Educational Studies in Mathematics*, vol.18, pp.371-397.
- Sierpinska, A. (1985) Obstacles épistémologiques relatifs à la notion de limite. *Recherches en Didactique des Mathématiques*, vol.6, no.1, pp.5-68.
- Shulman, L. (1987) Knowledge and Teaching: Foundations of the new reform. *Harvard Educational Review*, vol.57, no.1, pp.1-22.
- Shulman, L. S. (1986) Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, vol.15, no.2, pp.4-14.
- Silverman, R. (1969) *Modern Calculus and Analytic Geometry*, New York: The Macmillan Company.
- Steinbring, H. (1998) Elements of Epistemological Knowledge for Mathematics Teachers, *Journal of Mathematics Teacher Education*, vol.1, pp.157-189.
- Swokowski, E (1983) *Cálculo com Geometria Analítica*, São Paulo: Mac Graw Hill.
- Tall, D. (1992) The transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof, In: Grouws D. ed. *Handbook of Research on Mathematics Teaching and Learning*, New York : Mac Millan Publishing Company, pp.495-511.
- Tall, D. & Vinner, S. (1981) Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics* vol.12, pp.151-169.
- Trouche, L. (1996) *Etude des rapports entre processus de conceptualisation et processus d'instrumentalisation*, Thèse de doctorat de troisième cycle, Université de Montpellier: Laboratoire E.R.E.S.
- Yin, R. (1994) *Case Study Research: Design and methods*, 2<sup>nd</sup> edition, London: Sage Publications.
- Zack, V., Mousley J. & Breen, C. eds. (1997) *Developing Practice Teachers' Inquiry and Educational Change*. Geelong: Deakin University Press.